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Development Report

MEMBRANE AND EDGE EFFECT SOLUTIONS OF FILAMENT-WOUND SHELLS OF REVOLUTION

Ömer A. Fettahlioglu

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May 1969

ABSTRACT

A method for analyzing stresses and displacements in filament-wound shells of revolution is presented. This method of analysis is based on the membrane theory of shells but incorporates the edge effect theory to predict a more realistic behavior in the vicinity of polar openings. In addition, because it includes the contribution of the matrix, the present membrane theory is more accurate for the analysis of shells under nondestructive operating loads than is the so-called netting analysis which considers the filaments to be the only load-carrying members. The membrane theory is also modified to give accurate solutions in the shallow region of composite shells and indicates significant bending stresses due to orthotropy and the shape of the meridian. While these stresses are localized near the apex in thin shells, they may extend well down along the meridian in thick shells.

CONTENTS

List of Symbols

A_i	Cross-sectional area of fiber (in^2)
a_i	Thickness of equivalent solid sheet of filament (in)
C_{ij}	Moduli of elasticity of composite (lb/in^2)
d_i	Spacing of successive filaments in i-th layer (in)
E	Modulus of elasticity of isotropic material (lb/in^2)
E_f	Modulus of elasticity of filament (lb/in^2)
E_ℓ	Modulus of elasticity of liner (lb/in^2)
E_m	Modulus of elasticity of matrix (lb/in^2)
H	Radial stress resultant (lb/in)
h	Total thickness of shell (in)
h_f	Total thickness of equivalent solid sheets of filament (in)
h_ℓ	Liner thickness (in)
h_m	Matrix thickness (in)
M_θ	Bending moment in circumferential direction (in-lb/in)
M_ϕ	Bending moment in meridional direction (in-lb/in)
N	Number of filament circuits around shell
N_θ	Stress resultant in circumferential direction (lb/in)
N_ϕ	Stress resultant in meridional direction (lb/in)
$N_{\theta\phi}$	Shear stress resultant (lb/in)

n	Number of unidirectional equivalent solid sheet of filament
P	Axial line load at boundary (lb/in)
p	Uniform pressure (lb/in^2)
p_r	Applied load in normal direction (lb/in^2)
p_ϕ, p_s	Applied load in meridional direction (lb/in^2)
Q	Transverse shear stress resultant (lb/in)
R_o	Maximum value of r (in)
r	Radius of revolution of middle surface of shell (in)
$\tilde{r}$	Nondimensional radius of revolution (r/R_o)
r_1, r_2	Principal radii of curvature (in)
r_1', r_2'	Radii of curvature (in)
S	Nondimensional arc length (s/R_o)
S_{ij}	Moduli of compliance of composite (in^2/lb)
s	Arc length (in)
u	Meridional displacement (in)
V	Axial stress resultant (lb/in)
v	Circumferential displacement (in)
$\tilde{v}$	Axial displacement (in)
x	Argument of Lommel functions
z	Axis of revolution
α_i	Angle between filaments of i-th layer and parallel circle
β_i	Angle between filaments of i-th layer and meridian
γ	Angle between planar-wound filaments and axis of revolution
γ_1, γ_2	Curvilinear coordinates

δ	Radial displacement (in)
ϵ_x, ϵ_y	Extensional strains of middle surface in x and y directions, respectively
ϵ_{xy}	Shear strain of middle surface
$\tilde{\zeta}$	Distance along normal measured from reference surface (in)
θ	Circumferential angle
ν	Poisson's ratio of isotropic material
ν_ℓ	Poisson's ratio of liner
ν_m	Poisson's ratio of matrix
σ_f	Filament stress (lb/in^2)
σ_ℓ	Liner stress (lb/in^2)
σ_m	Matrix stress (lb/in^2)
ϕ	Angle between line normal to shell and axis of revolution
χ	Rotation of line normal to shell
Ω	Angle between curvilinear coordinates
ver and ver*	Real parts of Lommel functions
vei and vei*	Imaginary parts of Lommel functions

Subscripts

i	Layer number with reference to inner face of shell
r	Normal direction
θ	Circumferential direction
ϕ	Meridional direction

Superscripts

m	Membrane
*	Edge effect

MEMBRANE AND EDGE EFFECT SOLUTIONS OF FILAMENT-WOUND SHELLS OF REVOLUTION

Introduction

Filament-wound shells of revolution are widely used in modern technology as applied to aircraft, missile and spacecraft structures, pressure vessels, deep submergence vehicles and machine construction. They are formed by winding resin-coated high-strength filaments onto a mandrel and then heat-treating the resulting shell structure to produce a rigid composite.

The filaments are deposited in pairs that are symmetric with respect to a meridian, along lines defined by the intersection of the surface of the shell with a plane making an angle γ with the axis of revolution z. Such a winding pattern is referred to as planar-wound filamentary pattern. This technique of winding has found wide acceptance in the industry because it is easy to achieve in practice.

The present report describes an analytical method of stress analysis of filament-wound shells of revolution. This method of analysis is based on the membrane theory of shells but incorporates the edge effect theory to predict more realistic stresses in the vicinity of the polar opening where reinforcement devices and the shell, each of different elastic properties, interact. The local stresses near the line of distortion (i.e., the supported edge of the shell) are formulated on the basis of Gol'denveizer's edge effect theory (Ref. 1) and then subsequently reformulated with the use of Steele's class notes (Ref. 2) based on the method of asymptotic integration developed by Blumenthal (Ref. 3), for the purpose of correlating the consistency of the two methods. The two methods yield identical results if only the leading term of asymptotic solution is used.

A computer code based on the present theory has been developed A user's manual for the program including a study of the behavior of the filament-wound spherical shells will be published at a later date. The code will automatically plot the variation of the stresses along the meridian as well as through the thickness of the shell in the filament, matrix, and liner.

Region of Applicability of Membrane Theory

The membrane theory gives an approximately correct picture of the state of stress and deformation of a shell only at sufficient distances away from the lines of distortion of the state of stress on the middle surface.

The lines of distortion may be: 1) the edges of the shell; 2) lines along which discontinuities of the components of the external surface loads occur; 3) lines along which the middle surface of the shell has an abrupt change in curvature; 4) lines along which the rigidity of the shell or its thickness undergoes sudden changes.

When treating a shell which is not locally spherical and isotropic in the neighborhood of the apex, the membrane solutions are not generally valid. Recent work by C. R. Steele (Ref. 4) indicates that for orthotropic shells in the shallow region the particular solutions to the shell equations do not approach the "membrane" solutions. It is interesting to note that the particular solutions show that bending stresses will be present. These bending stresses are not due to edge effect solutions, but exist simply because of orthotropy and shape of the meridian curve.

Stress-Strain Relations for an Anisotropic Elastic Plate

The stress-strain relations for a general anisotropic material is of the form (Ref. 5)

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} \tag{1}$$

and its inverse

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{bmatrix} \tag{2}$$

where S_{ij} are the "moduli of compliance"; and C_{ij} are the "moduli of elasticity" of the anisotropic plate. They are related by the following expressions,

$$
\begin{aligned}
S_{11} &= \frac{C_{22}C_{33} - C_{23}^2}{\Delta} & S_{12} &= \frac{C_{13}C_{23} - C_{12}C_{33}}{\Delta} \\
S_{13} &= \frac{C_{12}C_{23} - C_{13}C_{22}}{\Delta} & S_{22} &= \frac{C_{11}C_{33} - C_{13}^2}{\Delta} \\
S_{23} &= \frac{C_{12}C_{13} - C_{11}C_{23}}{\Delta} & S_{33} &= \frac{C_{11}C_{22} - C_{12}^2}{\Delta}
\end{aligned}
\tag{3}
$$

$$\Delta = C_{11}C_{22}C_{33} + 2C_{12}C_{23}C_{13} - C_{11}C_{23}^2 + C_{22}C_{13}^2 + C_{33}C_{12}^2$$

If the lamina is orthotropic and the principal elastic axes coincide with the coordinate axes, $S_{13} = S_{23} = 0$
and the moduli of compliance S_{ij} can be expressed in terms of the elastic moduli and Poisson's ratio as follows

$$
\begin{aligned}
&S_{11} = \frac{1}{E_x}; \quad S_{12} = S_{21} = -\frac{\nu_{xy}}{E_x} = -\frac{\nu_{yx}}{E_y} \\
&S_{22} = \frac{1}{E_y}; \quad S_{33} = \frac{1}{G}
\end{aligned}
\tag{3a}
$$

Membrane Solution

The membrane theory formulated in this section is based on the assumption that both the resinous matrix and the filaments contribute to the strength of the composite shell. This present membrane theory is more realistic than the so-called netting analysis widely used in the industry, which is a

limiting case of the present theory. The netting analysis, based on the assumption that the filaments are the only load carrying components of the composite shell, is inadequate for furnishing the stresses developed in the shell under nondestructive operating loads. Neither is it capable of permitting determination of the membrane deformation, nor of permitting incorporation of refinements necessary for handling the shallow region of the shell because of orthotropic construction (Ref. 4), nor of permitting the formulation of edge effects.

For a shell of revolution subjected to loads having axial symmetry, the stress resultants N_ϕ and N_θ in meridional and circumferential directions, respectively (Figure 1), are

$$N_\phi^m = \frac{1}{r_2 \sin^2 \phi} \left[\int r_1 r_2 \left(p_r \cos\phi - p_\phi \sin\phi \right) \sin\phi \, d\phi + C \right]$$

$$N_\theta^m = p_r r_2 - N_\phi \frac{r_2}{r_1} \tag{4}$$

For uniformly distributed loading on a shell without polar opening, Eqs. (4) reduce to

$$N_\phi^m = \frac{pr_2}{2}$$

$$N_\theta^m = \frac{pr_2}{2}\left(2 - \frac{r_2}{r_1}\right) \tag{4a}$$

where the principal radii of curvature r_1 and r_2 are given by the formula

$$r_1 = \frac{\left[1 + \left(\frac{df(z)}{dz}\right)^2\right]^{3/2}}{\frac{d^2 f(z)}{dz^2}} \; ; \quad r_2 = f(z)\left[1 + \left(\frac{df(z)}{dz}\right)^2\right]^{1/2} \tag{5}$$

when the meridian curve of the shell of revolution is defined by a single equation

$$r = f(z) \tag{6}$$

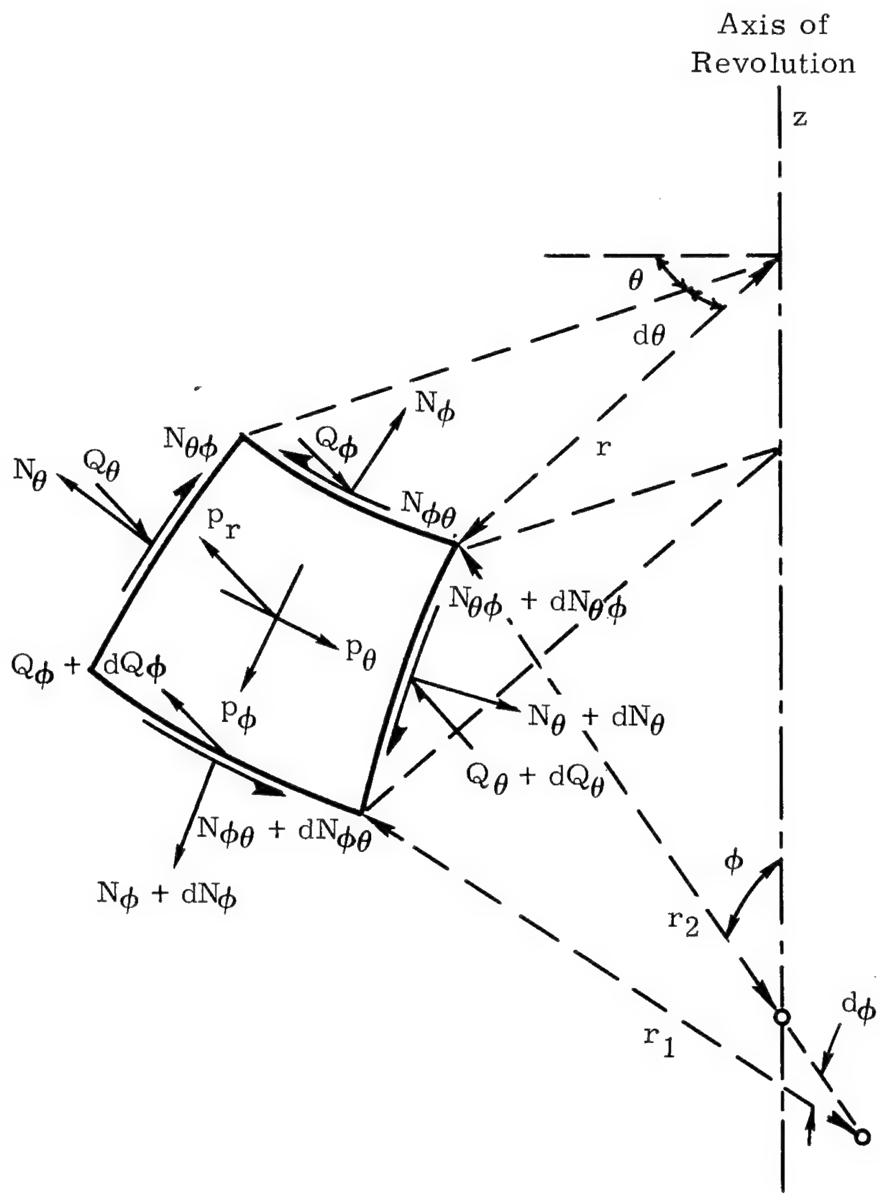


Figure 1. Stress Resultants and Applied Loads on Differential Element

If the shell is described by a pair of parametric equations

$$
\begin{aligned}
r &= r(s) \\
z &= z(s)
\end{aligned}
\tag{7}
$$

the radii of curvature can then be computed from the following relationships

$$
\begin{aligned}
r_1 &= \frac{\left[z' + (r')^2\right]^{3/2}}{z'r'' - r'z''} \\
r_2 &= \frac{r\left[(z')^2 + (r')^2\right]^{1/2}}{z'}
\end{aligned}
\tag{8}
$$

where the primes refer to the derivatives with respect to the parameter s which is defined herein as the arclength of the meridian curve.

For the given stress resultants N^m_θ, N^m_ϕ, $N^m_{\theta\phi}$, the mean stresses are

$$
\sigma_\theta = \frac{N^m_\theta}{h} \; ; \quad \sigma_\phi = \frac{N^m_\phi}{h} \; ; \quad \sigma_{\theta\phi} = \frac{N^m_{\theta\phi}}{h} \tag{9}
$$

where θ, ϕ are associated with x, y, respectively.

Since the filament patterns of composite shells of revolution are generated by the superposition of n/2 different pairs of layers which are symmetric with respect to the meridians, with $2A_i$ representing the cross-sectional area of the filaments of one distinct pair and d_i the spacing of the successive filaments, the ratio

$$
\frac{A_i}{d_i}
$$

represents the thickness a_i of an equivalent solid sheet of filament (Figure 2).

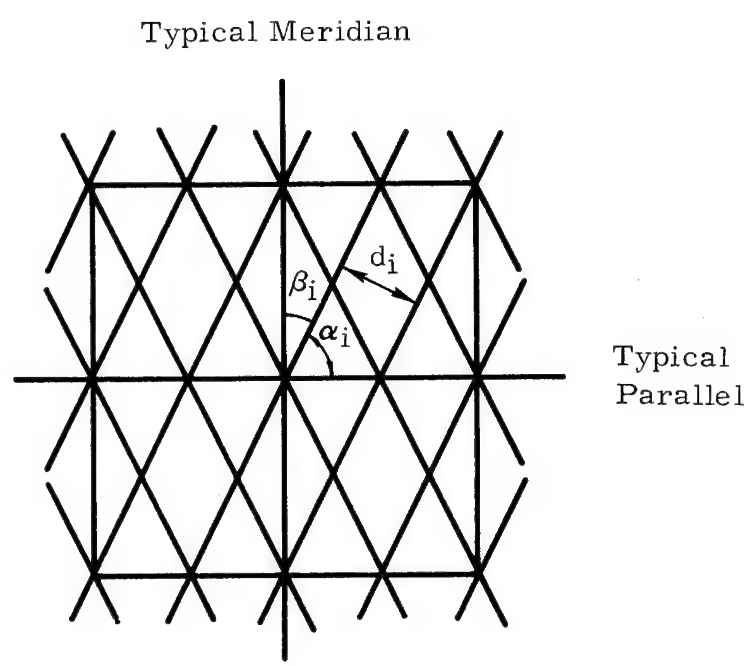


Figure 2. Element of Shell Illustrating Filament Pattern

Assuming that each filament crosses the equator and all other parallel circles of the shell an equal number of times, the spacing of the filaments along a parallel circle corresponding to a particular value of the position angle ϕ can be expressed as

$$d_i = \frac{2\pi r \sin\alpha_i}{N} \tag{10}$$

where, N is the number of circuits of filaments around the shell.

In order to present a method of analysis of greater generality, no specific filament pattern is assumed and thus the filament layers are variously oriented with reference to α_i.

Hence

$$a_i = \frac{NA_i}{2\pi r \sin\alpha_i} \tag{11}$$

where α_i is the angle that the filaments of the i-th layer makes with the parallel circle.

The total thickness of equivalent solid sheets of filament corresponding to a particular ϕ is

$$h_f = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{NA_i}{2\pi r \sin\alpha_i} \tag{12}$$

Hence, the net matrix thickness

$$h_m = h - h_f - h_\ell \tag{13}$$

h being the total thickness of the shell

h_ℓ being the thickness of the liner

For a unidirectional filamentary layer which is in a state of uniaxial tension, the linear strain ϵ_i takes the following form by the reduction of the Eqs. (A-9) with $\omega = \alpha_i$ (Ref. 6)

$$\epsilon_i = \epsilon_\theta \cos^2 \alpha_i + \epsilon_\phi \sin^2 \alpha_i + \epsilon_{\theta\phi} \cos \alpha_i \sin \alpha_i \tag{14}$$

The axial filament stress

$$\sigma_{f_i} = E_f \epsilon_i = E_f \left(\epsilon_\theta \cos^2 \alpha_i + \epsilon_\phi \sin^2 \alpha_i + \epsilon_{\theta\phi} \cos \alpha_i \sin \alpha_i \right) \tag{15}$$

The mean stress components σ_ϕ, σ_θ, $\sigma_{\theta\phi}$ will be obtained by combining the contributions of the force per unit width of the i-th filament layer (Figure 3), $\sigma_{f_i} \frac{a_i}{h} \cos^2 \alpha_i$, $\sigma_{f_i} \frac{a_i}{h} \sin^2 \alpha_i$, and $\sigma_{f_i} \frac{a_i}{h} \cos \alpha_i \sin \alpha_i$ respectively,

with those of the matrix and liner, and by summing over n filament layers,

$$\begin{aligned}
\sigma_\phi &= \frac{N_\phi^m}{h} = \sum_{i=1}^{n} \sigma_{f_i} \left(\frac{a_i}{h} \right) \sin^2 \alpha_i + \sigma_{\phi_m} \left(\frac{h_m}{h} \right) + \sigma_{\phi_\ell} \left(\frac{h_\ell}{h} \right) \\
\sigma_\theta &= \frac{N_\theta^m}{h} = \sum_{i=1}^{n} \sigma_{f_i} \left(\frac{a_i}{h} \right) \cos^2 \alpha_i + \sigma_{\theta_m} \left(\frac{h_m}{h} \right) + \sigma_{\theta_\ell} \left(\frac{h_\ell}{h} \right) \\
\sigma_{\theta\phi} &= \frac{N_{\theta\phi}^m}{h} = \sum_{i=1}^{n} \sigma_{f_i} \left(\frac{a_i}{h} \right) \cos \alpha_i \sin \alpha_i + \sigma_{\theta\phi_m} \left(\frac{h_m}{h} \right) + \sigma_{\theta\phi_\ell} \left(\frac{h_\ell}{h} \right)
\end{aligned} \tag{16}$$

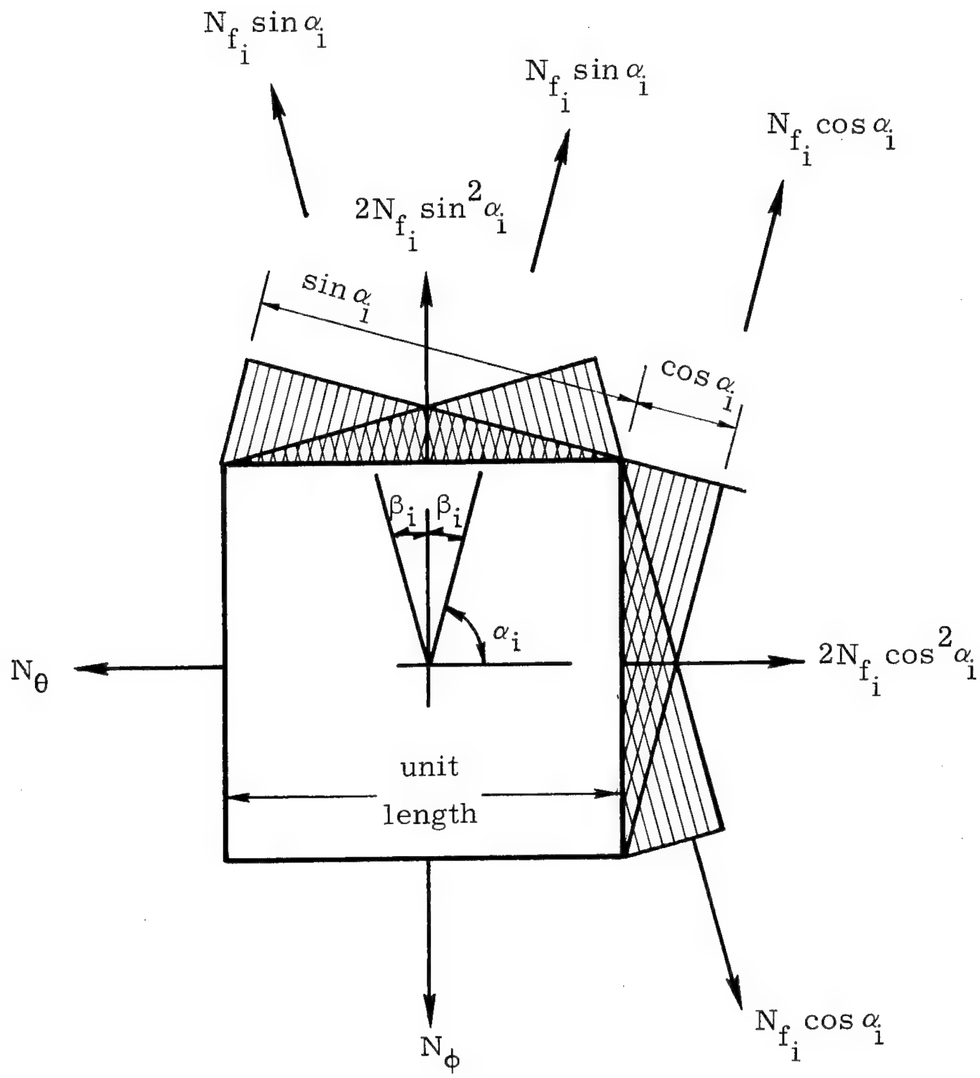


Figure 3. Forces in the i-th Filament Layer Pair

where the stress components in the matrix and the liner are given by

$$\sigma_{\phi_m} = \frac{E_m}{1 - \nu_m^2}\left(\epsilon_\phi + \nu_m \epsilon_\theta\right); \qquad \sigma_{\phi_\ell} = \frac{E_\ell}{1 - \nu_\ell^2}\left(\epsilon_\phi + \nu_\ell \epsilon_\theta\right)$$

$$\sigma_{\theta_m} = \frac{E_m}{1 - \nu_m^2}\left(\epsilon_\theta + \nu_m \epsilon_\phi\right); \qquad \sigma_{\theta_\ell} = \frac{E_\ell}{1 - \nu_\ell^2}\left(\epsilon_\theta + \nu_\ell \epsilon_\phi\right) \tag{17}$$

$$\sigma_{\theta\phi_m} = \frac{E_m}{2(1 + \nu_m)}\epsilon_{\theta\phi}; \qquad \sigma_{\theta\phi_\ell} = \frac{E_\ell}{2(1 + \nu_\ell)}\epsilon_{\theta\phi}$$

Substituting Eq. (15) and (17) into (16) yields

$$\begin{aligned}
\sigma_\phi &= \frac{E_f}{h}\sum_{i=1}^{n} a_i\left(\epsilon_\theta \cos^2\alpha_i \sin^2\alpha_i + \epsilon_\phi \sin^4\alpha_i + \epsilon_{\theta\phi}\cos\alpha_i \sin^3\alpha_i\right) \\
&+ \frac{h_m}{h}\frac{E_m}{\left(1 - \nu_m^2\right)}\left(\epsilon_\phi + \nu_m \epsilon_\theta\right) + \frac{h_\ell}{h}\frac{E_\ell}{1 - \nu_\ell^2}\left(\epsilon_\phi + \nu_\ell \epsilon_\theta\right)
\end{aligned}$$

$$\begin{aligned}
\sigma_\theta &= \frac{E_f}{h}\sum_{i=1}^{n} a_i\left(\epsilon_\theta \cos^4\alpha_i + \epsilon_\phi \cos^2\alpha_i \sin^2\alpha_i + \epsilon_{\theta\phi}\cos^3\alpha_i \sin\alpha_i\right) \\
&+ \frac{h_m}{h}\frac{E_m}{\left(1 - \nu_m^2\right)}\left(\epsilon_\theta + \nu_m \epsilon_\phi\right) + \frac{h_\ell}{h}\frac{E_\ell}{\left(1 - \nu_\ell^2\right)}\left(\epsilon_\theta + \nu_\ell \epsilon_\phi\right)
\end{aligned} \tag{18}$$

$$\begin{aligned}
\sigma_{\theta\phi} &= \frac{E_f}{h}\sum_{i=1}^{n} a_i\left(\epsilon_\theta \cos^3\alpha_i \sin\alpha_i + \epsilon_\phi \cos\alpha_i \sin^3\alpha_i + \epsilon_{\theta\phi}\cos^2\alpha_i \sin^2\alpha_i\right) \\
&+ \frac{h_m}{h}\frac{E_m}{2(1 + \nu_m)}\epsilon_{\theta\phi} + \frac{h_\ell}{h}\frac{E_\ell}{2(1 + \nu_\ell)}\epsilon_{\theta\phi}
\end{aligned}$$

Comparing Eq. (18) with Eq. (2) and associating θ and ϕ with x and y respectively

$$
\begin{aligned}
C_{11} &= \frac{E_f}{h} \sum_{i=1}^{n} a_i \cos^4 \alpha_i + \frac{h_m E_m}{h\left(1 - \nu_m^2\right)} + \frac{h_\ell E_\ell}{h\left(1 - \nu_\ell^2\right)} \\
C_{12} &= \frac{E_f}{h} \sum_{i=1}^{n} a_i \cos^2 \alpha_i \sin^2 \alpha_i + \frac{\nu_m}{1 - \nu_m^2} \frac{h_m E_m}{h} + \frac{\nu_\ell}{1 - \nu_\ell^2} \frac{h_\ell E_\ell}{h} \\
C_{13} &= \frac{E_f}{h} \sum_{i=1}^{n} a_i \cos^3 \alpha_i \sin \alpha_i \\
C_{22} &= \frac{E_f}{h} \sum_{i=1}^{n} a_i \sin^4 \alpha_i + \frac{h_m E_m}{h\left(1 - \nu_m^2\right)} + \frac{h_\ell E_\ell}{h\left(1 - \nu_\ell^2\right)} \\
C_{23} &= \frac{E_f}{h} \sum_{i=1}^{n} a_i \cos \alpha_i \sin^3 \alpha_i \\
C_{33} &= \frac{E_f}{h} \sum_{i=1}^{n} a_i \cos^2 \alpha_i \sin^2 \alpha_i + \frac{h_m E_m}{2h(1 + \nu_m)} + \frac{h_\ell E_\ell}{2h(1 + \nu_\ell)}
\end{aligned}
\tag{19}
$$

With C_{ij} determined, S_{ij} can be calculated from Eq. (3). The stress in the filaments of the i-th layer is obtained by substituting Eq. (1) into Eq. (15)

$$
\begin{aligned}
\sigma_{f_i} = E_f \Big\{ &\left[S_{11}\sigma_\theta + S_{12}\sigma_\phi + S_{13}\sigma_{\theta\phi}\right] \cos^2 \alpha_i + \left[S_{12}\sigma_\theta + S_{22}\sigma_\phi + S_{23}\sigma_{\theta\phi}\right] \sin^2 \alpha_i \\
&+ \left[S_{13}\sigma_\theta + S_{23}\sigma_\phi + S_{33}\sigma_{\theta\phi}\right] \cos \alpha_i \sin \alpha_i \Big\}
\end{aligned}
\tag{20}
$$

Rewriting in terms of the stress resultants,

$$\sigma_{f_i} = \frac{E_f}{h}\left\{N_\theta^m\left[S_{11}\cos^2\alpha_i + S_{12}\sin^2\alpha_i + S_{13}\cos\alpha_i \sin\alpha_i\right]\right.$$

$$+ N_\phi^m\left[S_{12}\cos^2\alpha_i + S_{22}\sin^2\alpha_i + S_{23}\cos\alpha_i \sin\alpha_i\right] \qquad (21)$$

$$\left. + N_{\theta\phi}^m\left[S_{13}\cos^2\alpha_i + S_{23}\sin^2\alpha_i + S_{33}\cos\alpha_i \sin\alpha_i\right]\right\}$$

Similarly, substituting Eq. (1) into Eq. (17), the components of stress in the matrix are obtained

$$\begin{bmatrix} \sigma_{\phi_m} \\ \sigma_{\theta_m} \\ \sigma_{\theta\phi_m} \end{bmatrix} = \frac{E_m}{h\left(1 - \nu_m^2\right)} \begin{bmatrix} (S_{12} + \nu_m S_{11}) & (S_{22} + \nu_m S_{12}) & (S_{23} + \nu_m S_{13}) \\ (S_{11} + \nu_m S_{12}) & (S_{12} + \nu_m S_{22}) & (S_{13} + \nu_m S_{23}) \\ \frac{(1 - \nu_m)}{2} S_{13} & \frac{(1 - \nu_m)}{2} S_{23} & \frac{(1 - \nu_m)}{2} S_{33} \end{bmatrix} \begin{bmatrix} N_\theta^m \\ N_\phi^m \\ N_{\theta\phi}^m \end{bmatrix} \qquad (22)$$

Likewise, the components of stress in the liner could be obtained by replacing the subscript "m" with "ℓ" in Eq. (22).

For symmetric filament pattern: $S_{13} = S_{23} = 0$

For symmetric filament pattern and symmetric loading: $S_{13} = S_{23} = N_{\theta\phi}^m = 0$

Limitations of the Membrane Theory

The membrane approximation to the particular solution of the shell equations is generally invalid when dealing with shallow shells which are non-isotropic and non-spherical in the vicinity of the apex. Prof. C. R. Steele of Stanford University in his recent work on orthotropic shells of revolution (Ref. 4) obtained a particular solution, in terms of Lommel's and Bessel's functions, which is equivalent to the well-known "membrane" solution in the steep portion of the shell but in the shallow portion gives significant bending stresses. These bending stresses are not due to edge effect solutions, but exist simply because of orthotropy and the shape of the meridian curve.

The bending stresses are confined to the immediate vicinity of the apex for thin shells; however, for thicker shells may extend well down along the meridian from the apex.

Expressions for the stress resultants, strains and deformations due to the particular solutions are as follows (Figure 4),

$$N_{\theta j} = \frac{N^m_{\theta j}}{\mu_j - 1} \operatorname{ver}^*_{\mu,\nu} x \qquad \text{for } \mu \neq 1 \text{ or } \mu \neq 5 \text{ with } j = 1, 2 \tag{23a}$$

$$N_{\theta 1} = N^m_{\theta 1} + \frac{3 p_r r}{4 \sin \phi} \operatorname{ver}^*_{1,\nu} x \qquad \text{for } \mu = 1, \; j = 1 \tag{23b}$$

$$N_{\theta 3} = N^m_{\theta 3} - \frac{P \cot \phi}{4 \pi s} \operatorname{ver}^*_{1,\nu} x \qquad \text{for } \mu = 1, \; j = 3 \tag{23c}$$

$$N_{\theta 1} = N^m_{\theta 1} + \frac{p_r r \cos^2 \phi}{\sin \phi} \left(\frac{\operatorname{ver}^*_{5,\nu} x}{4} - 1 \right) \qquad \text{for } \mu = 5, \; j = 1 \tag{23d}$$

$$N_{\theta 2} = N^m_{\theta 2} - p_s r \cos \phi \left(\frac{\operatorname{ver}^*_{5,\nu} x}{4} + \operatorname{ver}_{5,\nu} x - 1 \right) \qquad \text{for } \mu = 5, \; j = 2 \tag{23e}$$

$$N_{\phi j} = N^m_{\phi 1} \left\{ \cos^2 \phi_a \operatorname{ver}_{\mu_j,\nu} x + \sin^2 \phi_a \right\} \qquad \text{(all cases)} \tag{24}$$

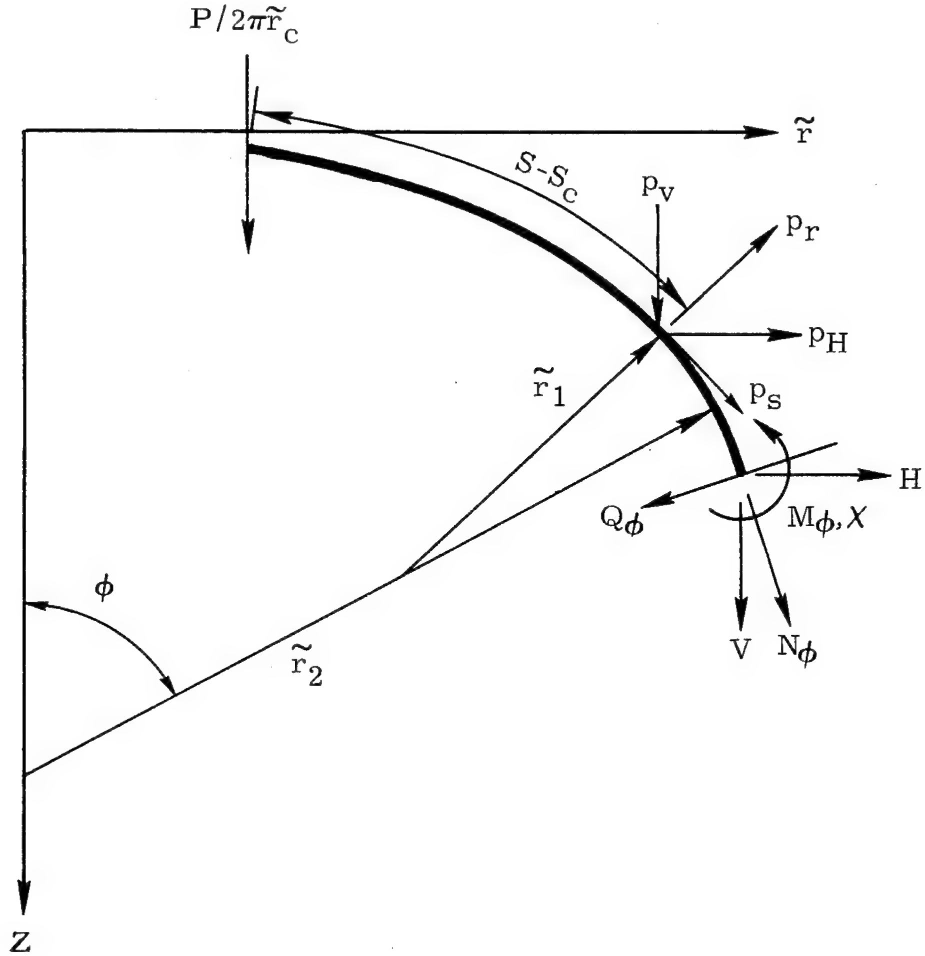


Figure 4. Shell Geometry

$$\chi = - \frac{r \cos \phi_a}{h^2 \sqrt{\frac{\Delta}{12}}} N^m_{\phi j} \, \mathrm{vei}_{\mu_j, \nu} x \qquad \text{(all cases)} \tag{25}$$

in which ϕ_a is the vertex angle for $b = 1$ (for $b \neq 1$, $\phi_a = 0$).

The stress couples are

$$M_{\theta j} = - \frac{h \cos \phi_a}{\sqrt{12 \Delta}} N^m_{\phi j} \left\{ C_{11} \cos \phi_a \, \mathrm{vei}_{\mu_j, \nu} x + C_{12} \frac{b}{2} \mathrm{vei}^*_{\mu_j, \nu} x \right\} \qquad \text{(all cases)}$$

$$M_{\phi j} = - \frac{h \cos \phi_a}{\sqrt{12 \Delta}} N^m_{\phi j} \left\{ C_{12} \cos \phi_a \, \mathrm{vei}_{\mu_j, \nu} x + C_{22} \frac{b}{2} \mathrm{vei}^*_{\mu_j, \nu} x \right\} \qquad \text{(all cases)} \tag{26}$$

where

$$N^m_{\phi j} = \frac{T_j}{r_2 \sin^2 \phi} \qquad j = 1, 2, 3 \tag{27}$$

$$N^m_{\theta j} = \begin{cases} pr_2 - \dfrac{T_j}{r_1 \sin^2 \phi} & j = 1 \\[2ex] - \dfrac{T_j}{r_1 \sin^2 \phi} & j = 2, 3 \end{cases} \tag{28}$$

$$\epsilon^m_{\theta j} = \frac{1}{h\Delta} \left(C_{22} N^m_{\theta j} - C_{12} N^m_{\phi j} \right) \qquad j = 1, 2, 3 \tag{29}$$

$$T_1 = \int_{s_c}^{s} p_r r \cos \phi \, ds \, , \quad T_2 = - \int_{s_c}^{s} p_s r \sin \phi \, ds, \quad T_3 = - \frac{P}{2\pi} \tag{30}$$

The argument of the Lommel functions is given by

$$x = \sqrt[4]{\frac{12\Delta R_o^2}{C_{22}^2 h^2}} \int_{S_c}^{S} \left(\frac{1}{\tilde{r}_2}\right)^{1/2} dS = \Lambda(S)i^{-3/2} \tag{31}$$

where

$$\Delta = C_{11}C_{22} - C_{12}^2 \tag{32}$$

and

$$\tilde{r} = \frac{r}{R_o}, \quad S = \frac{s}{R_o}, \quad Z = \frac{z}{R_o} \tag{33}$$

are the normalized distances with R_o = max r

$$\mu_1 = \frac{6 - b}{b}; \quad \mu_2 = \frac{4 + b}{b}; \quad \mu_3 = \frac{2 - b}{b} \tag{34}$$

$$\nu^2 = \frac{4}{b^2} \frac{C_{11}}{C_{22}} \tag{35}$$

$$Z = D\tilde{r}^b \left[1 + O(\tilde{r}^2)\right] \qquad \text{for } b \geqq 1 \text{ meridian curve} \tag{36}$$

The value

$b = 1$ gives the class of shells that are conical at the apex;

$b = 2$ gives the class of shells that are spherical at the apex;

and

$b > 2$ gives the class of shells with zero curvature at the apex.

The Lommel functions (Ref. 7) which are analogous to Thomson functions, expressed in terms of real and imaginary parts are as follows:

$$V_{\mu,\nu}\left(i^{3/2}x\right) = \mathrm{ver}_{\mu,\nu}\,x + i\,\mathrm{vei}_{\mu,\nu}\,x$$

$$V^{*}_{\mu,\nu}\left(i^{3/2}x\right) = \mathrm{ver}^{*}_{\mu,\nu}\,x + i\,\mathrm{vei}^{*}_{\mu,\nu}\,x \qquad (37)$$

$$\Lambda = i^{3/2}x$$

From the asymptotic behavior of $V_{\mu,\nu}(\Lambda)$ and $V^{*}_{\mu,\nu}(\Lambda)$ for large Λ one has, for fixed S while $k \to \infty$, the result

$$N_{\theta j} \sim N^{m}_{\theta j}, \quad N_{\phi j} \sim N^{m}_{\phi j}, \quad \epsilon_{\theta j} \sim \epsilon^{m}_{\theta j}$$

$$\chi_{j} \sim M_{\phi j} \sim M_{\theta j} \sim 0 \qquad (38)$$

The behavior of the Lommel functions for $\mu = 1$ and 2 is shown in Figures 5 and 6.

The results are uniformly valid throughout the shell and reduce to the shallow shell solutions for small Λ and approach non-shallow solutions asymptotically for $\Lambda \gg 1$.

Application

The results of the above formulation as applied to a shell of revolution with two clamped edges at $S = S_c$ and $S = S_d (S_d > S_c)$ to rigid rings can be advantageously represented in nondimensional curves for maximum stresses at the edges depending only on ν, b, C_{12}/C_{22} as parameters. These curves represent the behavior of shells of wide geometric variation. The curves presented for the various numerical values of the above parameters will enable the stress analyst to interpolate for his given shell configuration.

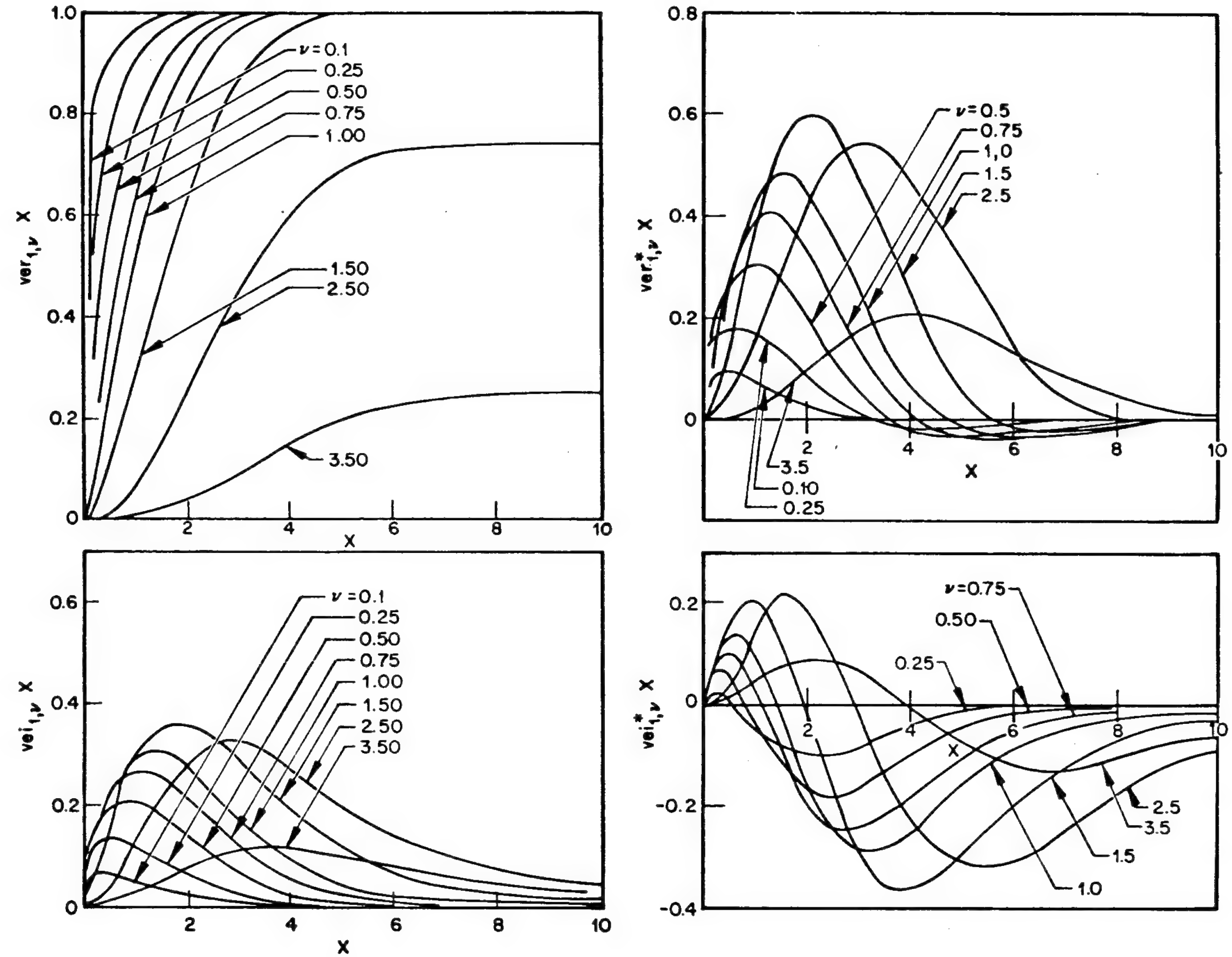


Fig. 5 The Functions $V_{1,\nu}(i^{3/2}x)$ and $V^*_{1,\nu}(i^{3/2}x)$

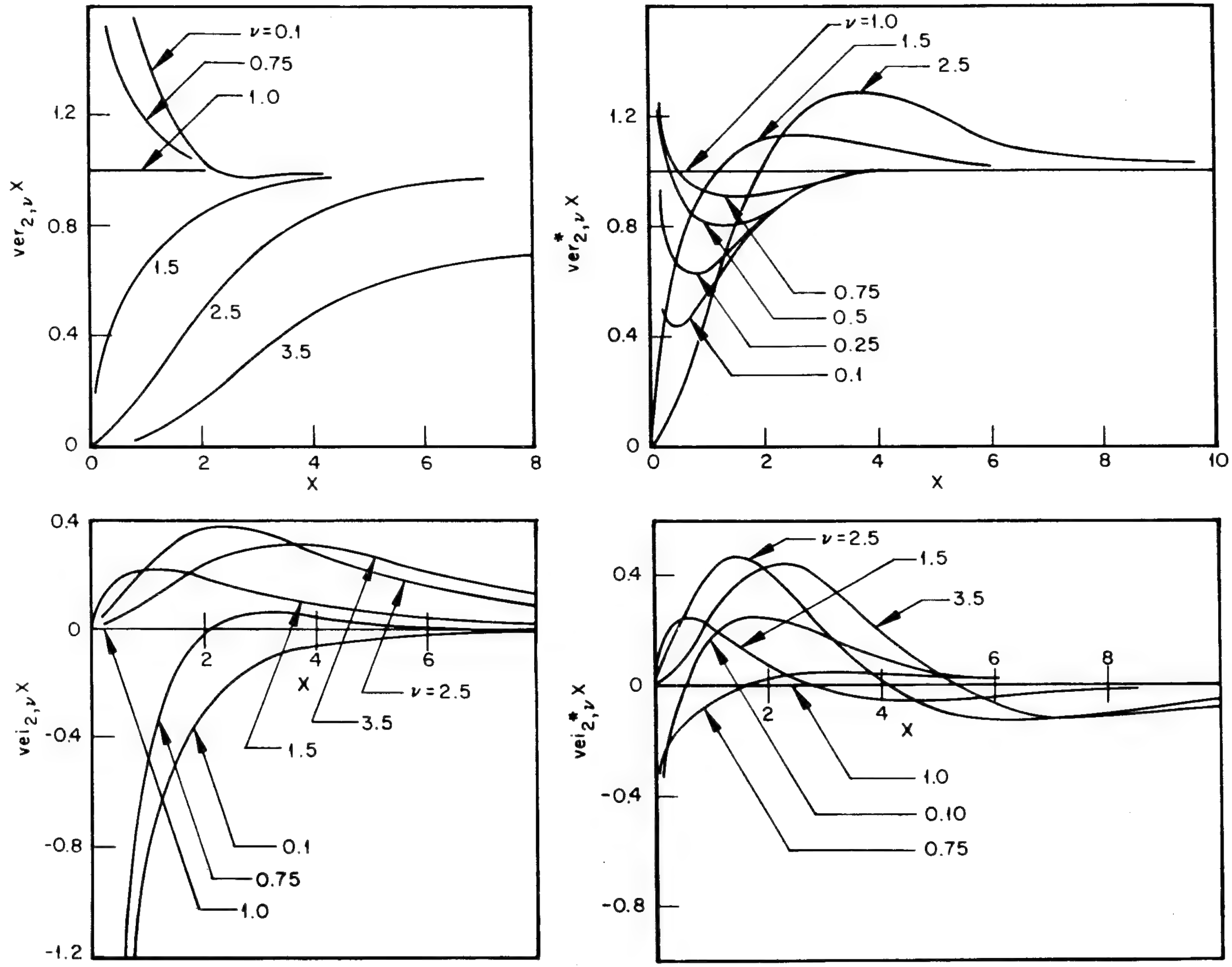


Fig. 6 The Functions $V_{2,\nu}(i^{3/2}x)$ and $V^*_{2,\nu}(i^{3/2}x)$

For the case of a spherical shell subjected to uniform pressure, p_r,

$b = 2$, $\mu = 2$, $\upsilon = \sqrt{\dfrac{C_{11}}{C_{22}}}$. The Figure 7 taken from Steele's paper (Ref. 4)

shows the nondimensional curves for $C_{12}/C_{22} = 1/10$, $1/3$, and $7/8$ with υ representing the family of curves. Curves with the M represent bending stresses, while those with N represent direct stresses.

The behavior of the filament-wound shell can be approximated by taking local values of x and υ along the meridian of the shell, Hartung (Ref. 8). This is equivalent to treating the shell as if it were composed of many segments each of which has constant x and υ, and then requiring the stress be continuous from one segment to the next. The x and υ, which are necessary to compute the Lommel functions are determined from expressions (31) & (35). The value of the argument of the Lommel functions depends upon the filament size and spacing, the moduli of elasticity of the filament, matrix and liner as well as the applied loading.

With the argument of the Lommel functions x, determined from expression (31); the modified N_θ and N_ϕ can be found from (23) and (24). These are then used with (21) and (22) to calculate the modified "membrane" solutions. The bending stresses produced because of orthotropy and the shape of the meridian curve can be calculated by substituting the values of the bending moments obtained from the equation (26) into equations (52) and (53).

Determination of the Winding Angle

The filaments are deposited by planar winding (Figure 8).

The winding angle β between the meridian and the filament is found from the definition

$$\cos\beta = \hat{t} \bullet \hat{T} = t_x T_x + t_y T_y + t_z T_z \tag{39}$$

where t and T are unit vectors tangent to the meridian and the filament curves, respectively. The components of these vectors can be obtained from the following expressions with primes denoting differentiation with respect to z,

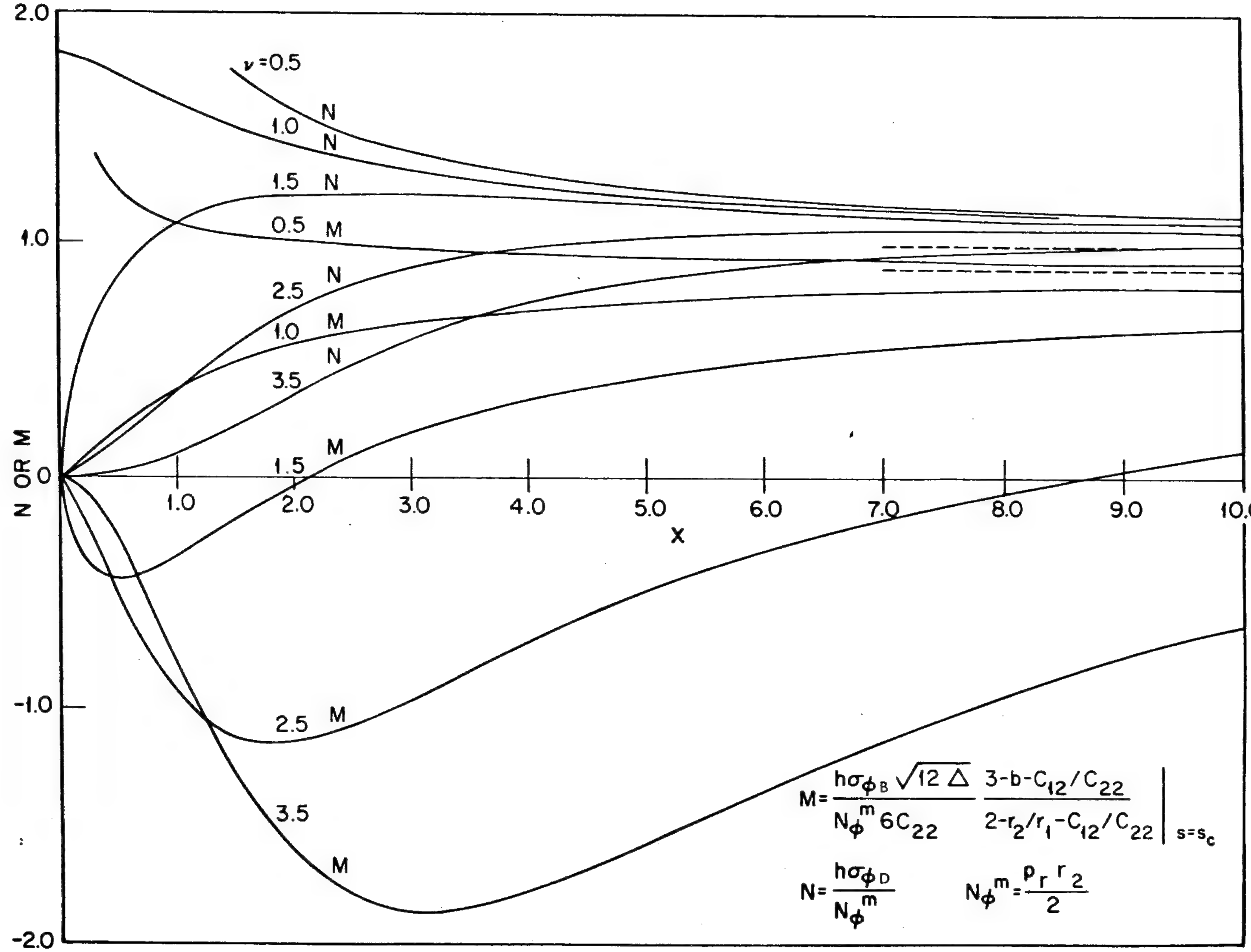


Fig. 7a Stress Resultants at Edge of Pressurized Dome Clamped to Rigid Ring $\left(\mu = 2, \frac{C_{12}}{C_{22}} = \frac{1}{10}\right)$

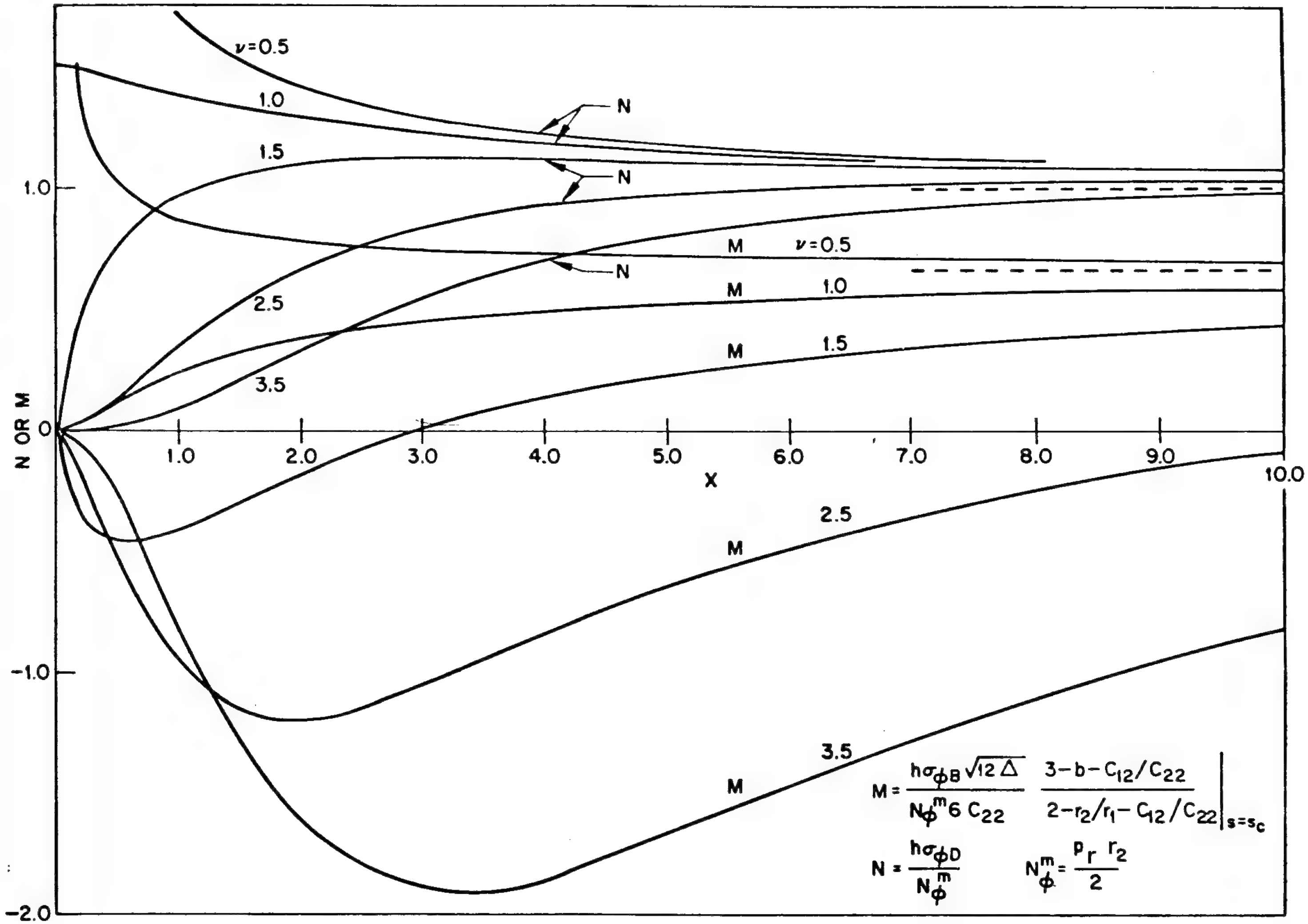


Fig. 7b Stress Resultants at Edge of Pressurized Dome Clamped to Rigid Ring $\left(\mu = 2, \frac{C_{12}}{C_{22}} = \frac{1}{3}\right)$

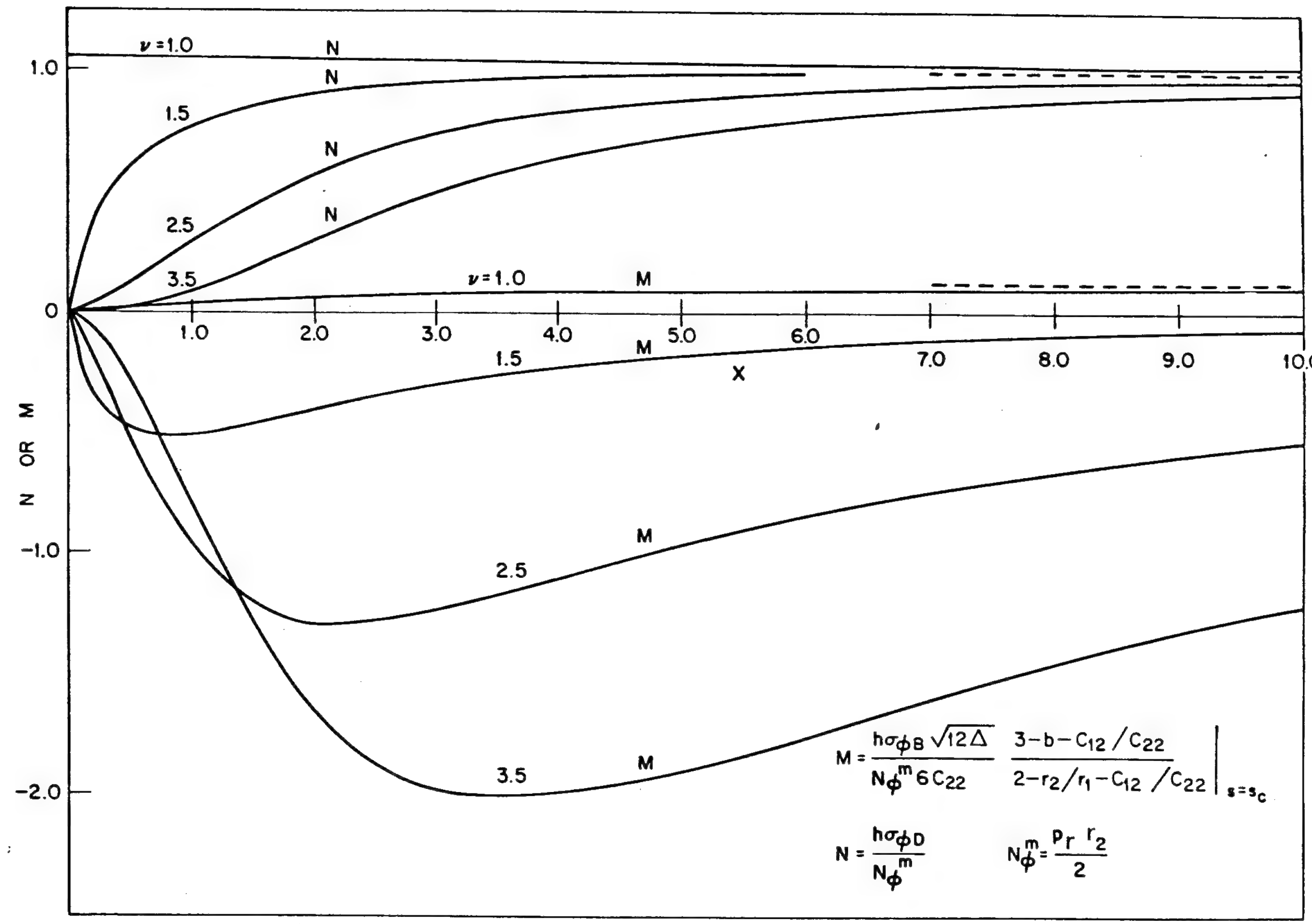


Fig. 7c Stress Resultants at Edge of Pressurized Dome Clamped to Rigid Ring $\left(\mu = 2, \frac{C_{12}}{C_{22}} = \frac{7}{8}\right)$

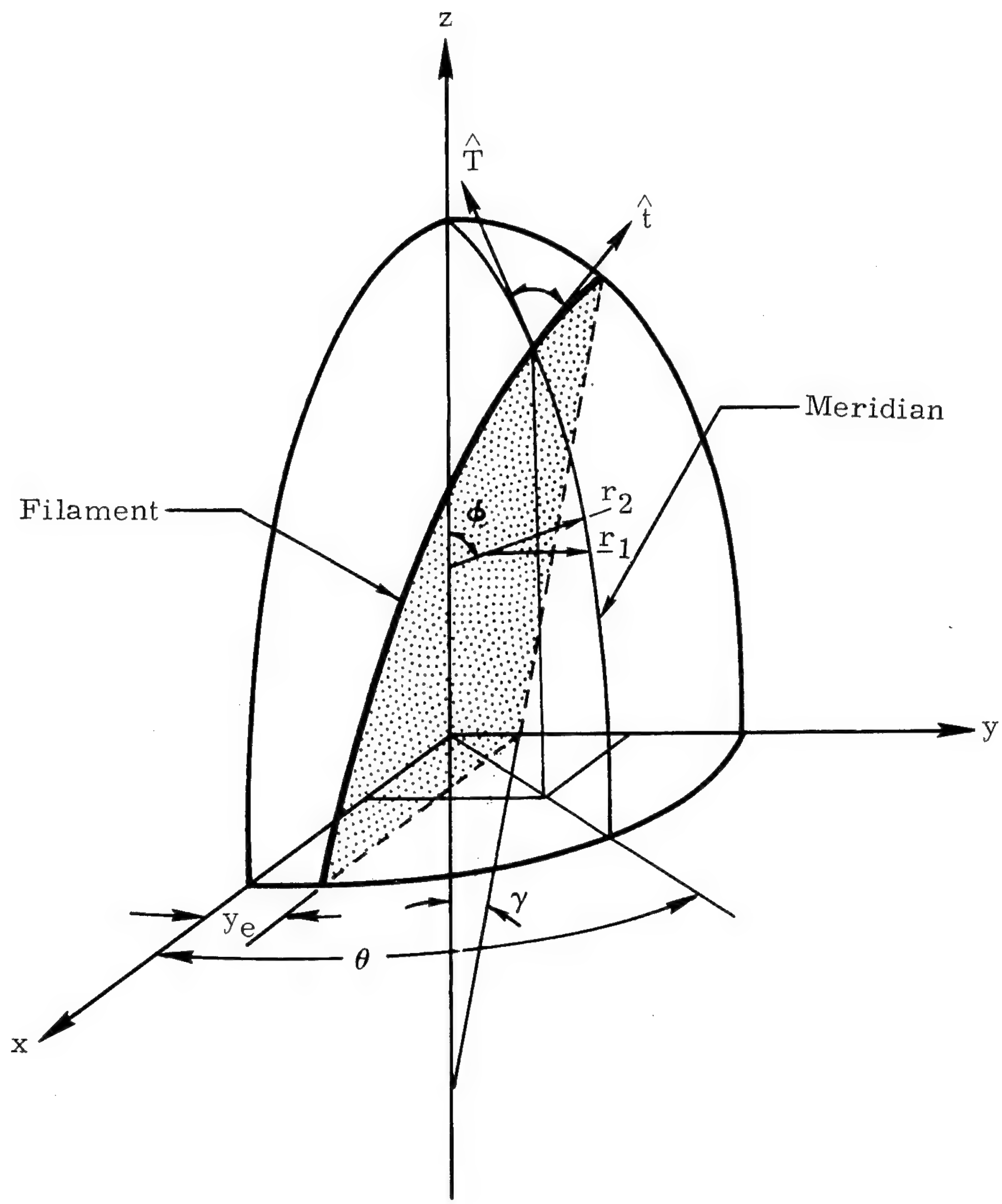


Figure 8. Geometry of Planar-Wound Shell of Revolution

$$t_x = \cos(\hat{t}, x) = \frac{x'}{\sqrt{x'^2 + y'^2 + 1}} \;; \quad T_x = \cos(\hat{T}, x) = \frac{r' \cos\theta}{\sqrt{1 + r'^2}}$$

$$t_y = \cos(\hat{t}, y) = \frac{y'}{\sqrt{x'^2 + y'^2 + 1}} \;; \quad T_y = \cos(\hat{T}, y) = \frac{r' \sin\theta}{\sqrt{1 + r'^2}} \tag{40}$$

$$t_z = \cos(\hat{t}, z) = \frac{1}{\sqrt{x'^2 + y'^2 + 1}} \;; \quad T_z = \cos(\hat{T}, z) = \frac{1}{\sqrt{1 + r'^2}}$$

The projection of the filament curve on the (y, z) plane is given by

$$x = \sqrt{r^2 - (y_e + z \tan\gamma)^2} \tag{41}$$

where y_e is the offset of the winding plane. R. Hartung (Ref. 9) obtained a general expression for the winding angle β as follows

$$\tan^2\beta = \frac{\left[rr' - (y_e + z\tan\gamma)\tan\gamma\right]^2 + (\tan^2\gamma - r'^2)\left[r^2 - (y_e + z\tan\gamma)^2\right]}{(1 + r'^2)\left[r^2 - (y_e + z\tan\gamma)^2\right]} \tag{42}$$

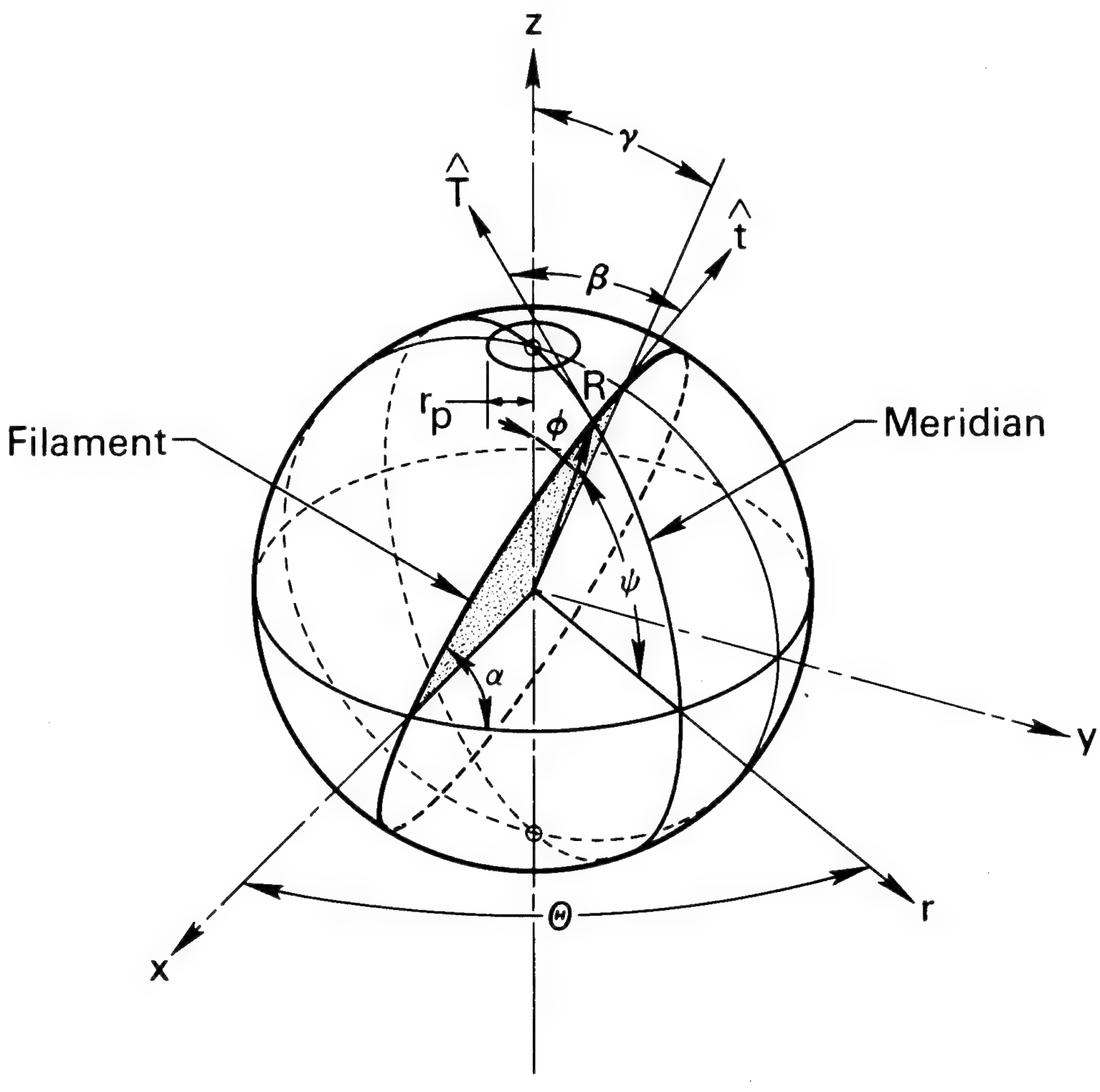
For the special case of a spherical shell (Figure 9) wound along its geodesics, by substituting the following expressions

$$r = \sqrt{R^2 - z^2}$$

$$r = R\cos\psi \qquad r' = \frac{dr}{dz} = -\frac{z}{r} = -\tan\psi \tag{43}$$

$$z = R\sin\psi$$

also $y_e = 0$



$$\sin\beta = \frac{\sin\gamma}{\sin\phi}$$

Figure 9. Planar-Filament-Wound Spherical Shell

the equation (42) becomes

$$\tan^2\beta = \frac{[R\cos\psi(-\tan\psi) - (R\sin\psi\tan\gamma)\tan\gamma]^2 + (\tan^2\gamma - \tan^2\psi)[R^2\cos^2\psi - (R\sin\psi\tan\gamma)^2]}{(1+\tan^2\psi)[R^2\cos^2\psi - (R\sin\psi\tan\gamma)^2]} \qquad (42a)$$

which reduces to a simple relationship among the three spatial angles β, ϕ, γ

$$\sin\beta = \frac{\sin\gamma}{\sin\phi} \qquad \text{where } \phi = \frac{\pi}{2} - \psi \qquad (43)$$

Bending Stresses

Assuming that the shell thickness is made up from a large number of layers of the same elastic properties and further assuming that the variation of the moduli of compliance S_{ij} along the meridian of the shell is slow, for unsymmetric loading (Ref. 6),

$$\begin{aligned} \epsilon_x &= \epsilon_\theta = \tilde{\zeta}\kappa_\theta \\ \epsilon_y &= \epsilon_\phi = \tilde{\zeta}\kappa_\phi \\ \epsilon_{xy} &= \epsilon_{\theta\phi} = \tilde{\zeta}\kappa_{\theta\phi} \end{aligned} \qquad (44)$$

where the changes in the meridional and the circumferential curvatures and the twist

$$\begin{aligned} \kappa_\phi &= \frac{\partial \chi_\phi}{\partial s}; \quad \kappa_\theta = \frac{1}{r}\left(\frac{\partial \chi_\theta}{\partial \theta} + \chi_\phi \cos\phi\right) \\ \kappa_{\theta\phi} &= \frac{r}{r_1}\frac{\partial}{\partial\phi}\left(\frac{\chi_\theta}{r}\right) + \frac{1}{r}\frac{\partial \chi_\phi}{\partial \theta} \end{aligned} \qquad (45)$$

the stresses are obtained from Equation (2)

$$\begin{aligned}
\sigma_\theta &= \tilde{\zeta}(C_{11}\kappa_\theta + C_{12}\kappa_\phi + C_{13}\kappa_{\theta\phi}) \\
\sigma_\phi &= \tilde{\zeta}(C_{12}\kappa_\theta + C_{22}\kappa_\phi + C_{23}\kappa_{\theta\phi}) \\
\sigma_{\theta\phi} &= \tilde{\zeta}(C_{13}\kappa_\theta + C_{23}\kappa_\phi + C_{33}\kappa_{\theta\phi})
\end{aligned} \tag{46}$$

The stress couples (Figure 10)

$$M_\theta,\ M_\phi,\ M_{\theta\phi} = \int_{-h/2}^{h/2} [\sigma_\theta,\ \sigma_\phi,\ \sigma_{\theta\phi}]\tilde{\zeta}\, d\tilde{\zeta} \tag{47}$$

yield

$$\begin{bmatrix} M_\theta \\ M_\phi \\ M_{\theta\phi} \end{bmatrix} = \frac{h^3}{12}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\begin{bmatrix} \kappa_\theta \\ \kappa_\phi \\ \kappa_{\theta\phi} \end{bmatrix} \tag{48}$$

In symbolic form

$$[M] = \frac{h^3}{12}\,[C]\,[\kappa] \tag{48a}$$

$$[\kappa] = \left(\frac{12}{h^3}\right)[C]^{-1}\,[M] \tag{49}$$

where $[C]^{-1}$ is the inverse of $[C]$ and is given by

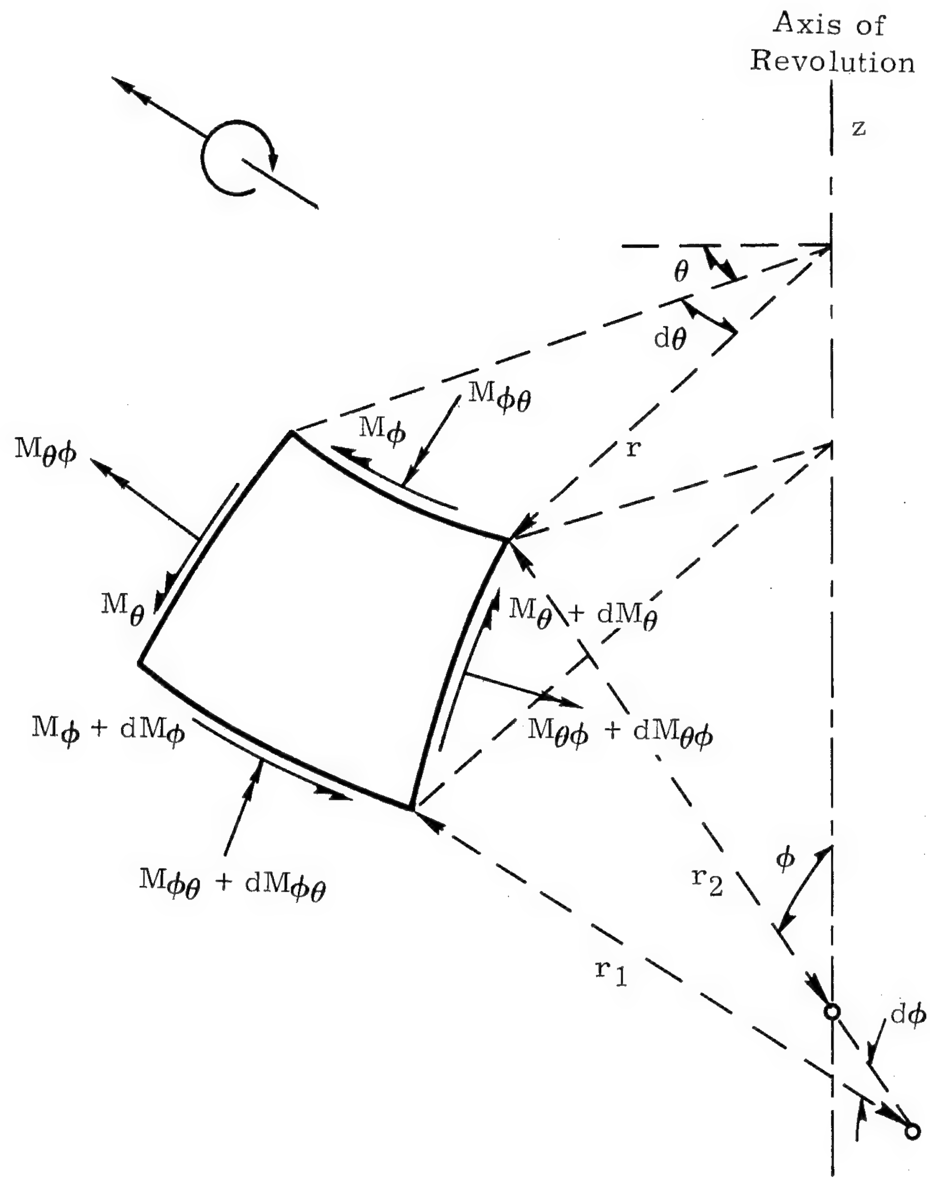


Figure 10. Stress Couples Acting on Differential Element

$$
[C]^{-1} = \frac{1}{\Delta_c}\begin{bmatrix} \left(C_{22}C_{33} - C_{23}^2\right) & \left(C_{13}C_{23} - C_{12}C_{33}\right) & \left(C_{12}C_{23} - C_{13}C_{22}\right) \\ & \left(C_{11}C_{33} - C_{13}^2\right) & \left(C_{12}C_{13} - C_{11}C_{23}\right) \\ & & \left(C_{11}C_{22} - C_{12}^2\right) \end{bmatrix} \quad (50)
$$

where

$$
\Delta_c = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix} \quad (50a)
$$

Using expression (3) in Equation (49) and substituting in turn in Equation (44)

$$
\begin{bmatrix} \epsilon_\theta \\ \epsilon_\phi \\ \epsilon_{\theta\phi} \end{bmatrix} = \tilde{\zeta}\left(\frac{12}{h^3}\right)\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}\begin{bmatrix} M_\theta \\ M_\phi \\ M_{\theta\phi} \end{bmatrix} \quad (51)
$$

Substituting in Equation (15), the axial filament stress in obtained.

$$
\begin{aligned}
\sigma_{f_i} = \tilde{\zeta}\left(\frac{12}{h^3}\right) E_f \Big\{ & M_\theta\left(S_{11}\cos^2\alpha_i + S_{12}\sin^2\alpha_i + S_{13}\cos\alpha_i \sin\alpha_i\right) \\
& + M_\phi\left(S_{12}\cos^2\alpha_i + S_{22}\sin^2\alpha_i + S_{13}\cos\alpha_i \sin\alpha_i\right) \quad (52) \\
& + M_{\theta\phi}\left(S_{13}\cos^2\alpha_i + S_{23}\sin^2\alpha_i + S_{33}\cos\alpha_i \sin\alpha_i\right)\Big\}
\end{aligned}
$$

Similarly, Equation (17) yields the stresses for the matrix

$$\sigma_{\phi_m} = \tilde{\zeta}\left(\frac{12}{h^3}\right)\left(\frac{E_m}{1-\nu_m^2}\right)\Big[M_\theta\left(S_{12}+\nu_m S_{11}\right)+M_\phi\left(S_{22}+\nu_m S_{12}\right) + M_{\theta\phi}\left(S_{23}+\nu_m S_{13}\right)\Big]$$

$$\sigma_{\theta_m} = \tilde{\zeta}\left(\frac{12}{h^3}\right)\left(\frac{E_m}{1-\nu_m^2}\right)\Big[M_\theta\left(S_{11}+\nu_m S_{12}\right)+M_\phi\left(S_{12}+\nu_m S_{22}\right) + M_{\theta\phi}\left(S_{13}+\nu_m S_{23}\right)\Big] \tag{53}$$

$$\sigma_{\theta\phi_m} = \tilde{\zeta}\left(\frac{12}{h^3}\right)\frac{E_m}{2(1+\nu_m)}\left(S_{13}M_\theta + S_{23}M_\phi + S_{33}M_{\theta\phi}\right)$$

For symmetric winding and unsymmetric loading: $S_{13} = S_{23} = 0$
For symmetric winding and symmetric loading: $S_{13} = S_{23} = M_{\theta\phi} = 0$

Similarly, the components of stress in the liner are obtained by replacing the subscript "m" with "ℓ" in Equation (53).

The bending stresses can thus be computed in terms of the stress couples M_θ, M_ϕ and the twisting moment $M_{\theta\phi}$ applied on the edges of a shell element. For the edge effect behavior of a shell, the twisting moment $M_{\theta\phi}$ plays a secondary role and may be neglected; however, the stress couples play an essential role. The principal one of them is M_ϕ. The stress couples will be furnished in the section "Edge Effect Solutions" in terms of the membrane stress resultants, N_θ^m and N_ϕ^m.

Reduction of the Equilibrium Equations to Two Differential Equations of the Second Order

Denoting the angle of rotation of a tangent to a meridian by χ

$$\chi = -\frac{dw}{ds} + \frac{u}{r_1} = \frac{1}{r_1}\left(-\frac{dw}{d\phi} + u\right) \tag{54}$$

and introducing $U = r_2 Q_\phi$, considerable simplification can be achieved in the following formulation (Ref.10).

In the absence of surface loading, the equilibrium equations become,

$$\begin{aligned} N_\phi &= Q_\phi \cot\phi \\ r_1 N_\theta + r_2 N_\phi &= \frac{1}{\sin\phi}\frac{d(rQ_\phi)}{d\phi} \end{aligned} \tag{55}$$

$$\begin{aligned} N_\phi &= \frac{1}{r_2} U \cot\phi \\ N_\theta &= \frac{1}{r_1}\frac{dU}{d\phi} \end{aligned} \tag{56}$$

For axisymmetrical loads, with the substitution of the strain components of the middle surface in terms of the deformations,

$$\epsilon_\phi = \frac{1}{r_1}\left(w + \frac{du}{d\phi}\right) ; \qquad \epsilon_\theta = \frac{1}{r_2}(w + u\cot\phi) \tag{57}$$

the stress resultants and stress couples become,

$$\begin{aligned} N_\theta &= h\left[\frac{C_{11}}{r_2}(w + u\cot\phi) + \frac{C_{12}}{r_1}\left(w + \frac{du}{d\phi}\right)\right] \\ N_\phi &= h\left[\frac{C_{12}}{r_2}(w + u\cot\phi) + \frac{C_{22}}{r_1}\left(w + \frac{du}{d\phi}\right)\right] \end{aligned} \tag{58}$$

$$M_\theta = \frac{h^3}{12}\left[C_{11}\frac{1}{r_2}\chi \cot\phi + \frac{C_{12}}{r_1}\frac{d\chi}{d\phi}\right]$$

$$M_\phi = \frac{h^3}{12}\left[C_{12}\frac{1}{r_2}\chi \cot\phi + \frac{C_{22}}{r_1}\frac{d\chi}{d\phi}\right] \tag{59}$$

To establish the first equation connecting χ and U, the Equation (58) gives,

$$w + \frac{du}{d\phi} = \frac{r_1}{h}\frac{1}{\left(C_{12}^2 - C_{11}C_{22}\right)}\left(N_\theta C_{12} - N_\phi C_{11}\right) \tag{60a}$$

$$w + u\cot\phi = \frac{r_2}{h}\frac{1}{\left(C_{11}C_{22} - C_{12}^2\right)}\left(N_\theta C_{22} - N_\phi C_{12}\right) \tag{60b}$$

Eliminating w from these equations, and using expressions (3) for the moduli of compliance S,

$$\frac{du}{d\phi} - u\cot\phi = \frac{1}{h}\left[(r_1 S_{22} - r_2 S_{12})\, N_\phi - (r_2 S_{11} - r_1 S_{12})\, N_\theta\right] \tag{61}$$

Differentiating Equation (60b)

$$\frac{du}{d\phi}\cot\phi - \frac{u}{\sin^2\phi} + \frac{dw}{d\phi} = \frac{d}{d\phi}\left[\frac{r_2}{h}\left(S_{11}N_\theta + S_{12}N_\phi\right)\right] \tag{62}$$

The derivative of u can be readily eliminated from Equations (61) and (62) to yield

$$\frac{dw}{d\phi} - u = \frac{d}{d\phi}\left[\frac{r_2}{h}\left(S_{11}N_\theta + S_{12}N_\phi\right)\right] - \frac{\cot\phi}{h}\left[(r_1 S_{22} - r_2 S_{12})\, N_\phi - (r_2 S_{11} - r_1 S_{12})\, N_\theta\right]$$

Substituting expressions (56) for N_ϕ and N_θ, an equation relating U and χ is finally obtained.

$$
\begin{aligned}
&\frac{S_{11}}{h}\left(\frac{r_2}{r_1}\right)\frac{d^2U}{d\phi^2} + \frac{dU}{d\phi}\left[\frac{S_{11}}{h}\frac{d}{d\phi}\left(\frac{r_2}{r_1}\right) + \frac{dS_{11}}{d\phi}\left(\frac{r_2}{r_1}\right)\frac{1}{h} + \frac{r_2}{r_1}S_{11}\frac{d}{d\phi}\left(\frac{1}{h}\right) + \frac{S_{12}}{h}\cot\phi\right.\\
&\qquad \left. + \frac{\cot\phi}{hr_1}(r_2S_{11} - r_1S_{12})\right] + U\left[\frac{\cot\phi}{h}\frac{dS_{12}}{d\phi} + S_{12}\cot\phi\frac{d}{d\phi}\left(\frac{1}{h}\right)\right.\\
&\qquad \left. - \frac{1}{\sin^2\phi}\frac{S_{12}}{h} - \frac{\cot^2\phi}{hr_2}(r_1S_{22} - r_2S_{12})\right] = -r_1\chi
\end{aligned}
\tag{63}
$$

For the isotropic case:

$$
\begin{aligned}
C_{11} &= C_{22} = \frac{E}{1-\nu^2}\ ; \quad C_{12} = E\frac{\nu}{1-\nu^2}\\
S_{11} &= S_{22} = \frac{1}{E}\ ; \quad S_{12} = -\frac{\nu}{E}
\end{aligned}
\tag{64}
$$

and hence the Equation (63) reduces to Equation (315) of Timoshenko (Ref.10).

The second equation for U and χ is obtained by substituting expressions(59) for M_θ and M_ϕ in the following moment equilibrium equation,

$$
\frac{d}{ds}(rM_\phi) - M_\theta\cos\phi - rQ_\phi = 0 \tag{65}
$$

which in turn yields

$$
\begin{aligned}
&\frac{r_2}{r_1^2}\frac{d^2\chi}{d\phi^2} + \frac{1}{r_1}\frac{d\chi}{d\phi}\left[\cot\phi + r_2\frac{d}{d\phi}\left(\frac{1}{r_1}\right) + \frac{r_2}{r_1}\frac{3}{h}\frac{dh}{d\phi}\right] + \frac{C_{12}}{C_{22}}\frac{\chi}{r_1}\left[\left(1 - \frac{C_{11}}{C_{12}}\right)\frac{r_1}{r_2}\cot^2\phi\right.\\
&\qquad \left. + r_2\cot\phi\frac{d}{d\phi}\left(\frac{1}{r_2}\right) - \frac{1}{\sin^2\phi} + \frac{3}{h}\cot\phi\frac{dh}{d\phi}\right] = \frac{U}{C_{22}}\left(\frac{12}{h^3}\right)
\end{aligned}
\tag{66}
$$

The Equation (66) reduces to Equation (316) of Timoshenko (Ref. 8) for the case of isotropy.

Thus, the problem of bending of a shell of revolution under stress resultants and couples along its edge, a parallel circle, is reduced to the integration of Equations (63) and (66). Since the stresses produced by the stress resultants and couples applied at the edge of the shell, present themselves as a rapidly decaying state of stress when the angle ϕ is not too small, a simplified solution can be obtained by means of Geckeler approximations applied to spherical shells. Neglecting the functions U and χ and their first derivatives in comparison with the second derivatives, the Equations (63) and (66) become,

$$\frac{d^2U}{d\phi^2} = -\frac{h}{S_{11}} \frac{r_1^2}{r_2} \chi \tag{63a}$$

$$\frac{d^2\chi}{d\phi^2} = \frac{12}{h^3} \frac{r_1^2}{r_2} \frac{\left(S_{11}S_{22} - S_{12}^2\right)}{S_{11}} U \tag{66a}$$

These equations can be combined into a single differential equation.

$$\frac{d^4\chi}{d\phi^4} + 4k^4\chi = 0 \tag{67}$$

$$k^4 = \frac{3}{h^2}\left(\frac{r_1^2}{r_2}\right)^2 \frac{\left(S_{11}S_{22} - S_{12}^2\right)}{S_{11}^2} \tag{68}$$

Here k is a function of ϕ angle. A satisfactory approximate solution of Equation (67) can be obtained by replacing $k(\phi)$ by a certain constant average value as suggested by Timoshenko (Ref. 10).

Edge Effect Solutions

Every state of stress arising in a shell as a consequence of the fact that it has a line of distortion is called an edge effect. The lines of distortion may be: 1) the edges of the shell; 2) lines along which discontinuities of the components of the external surface loads occur; 3) lines along which the curvature of the middle surface changes abruptly; 4) lines along which the rigidity of the shell or its thickness undergoes sudden changes.

A complete study of the state of stress consists of the membrane analysis of the shell and a determination of the local stresses occurring near lines of distortions due to edge effects. The membrane analysis gives an approximately correct picture of the state of stress and deformation of the shell only at sufficient distances away from the lines of distortion. The edge effect presents itself as a rapidly decaying state of stress away from the edge of the shell.

A. Gol'denveizer's Solution

Assuming a system of curvilinear coordinates γ_1, and γ_2 has been constructed on the shell (Figure 11), in which the line of distortion

$$\gamma_1 = \gamma_{10} = \text{constant}$$

Gol'denveizer obtained (Ref. 1).

$$\frac{1}{A^4}\frac{d^4\chi}{d\gamma_1^4} + \frac{12(1-\nu^2)}{h^2 r_2'^2}\chi = 0 \tag{69}$$

The quantities r_1' and r_2' represent the radii of curvature of the normal sections, cut along the coordinate lines.

First fundamental form of the theory of surfaces is

$$(ds)^2 = A^2(d\gamma_1)^2 + 2AB\cos\Omega\, d\gamma_1 d\gamma_2 + B^2(d\gamma_2)^2 \tag{70}$$

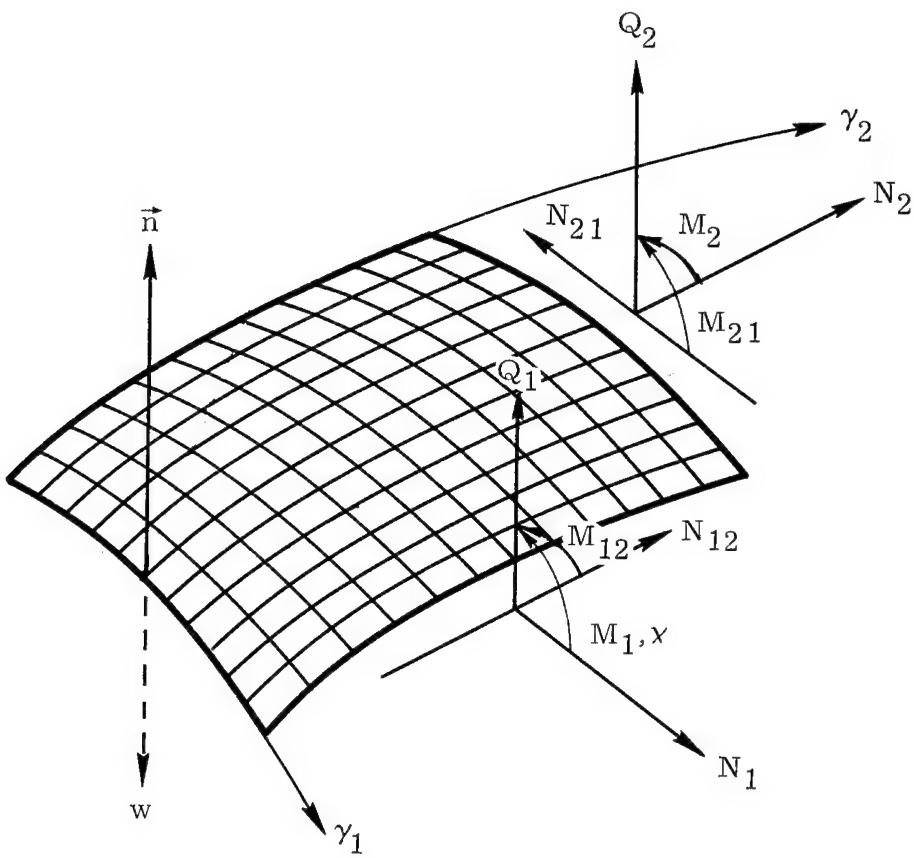


Figure 11. Differential Surface Element in Curvilinear Coordinates

where

$$\frac{\vec{r}_{o,\gamma_1}}{A} \cdot \frac{\vec{r}_{o,\gamma_2}}{B} = \cos\Omega ; \qquad A = \left|\frac{\partial \vec{r}_o}{\partial \gamma_1}\right| ; \qquad B = \left|\frac{\partial \vec{r}_o}{\partial \gamma_2}\right| \tag{71}$$

$\vec{r}_0$ is the position vector, $\vec{r}_0(\gamma_1, \gamma_2)$

Ω is the angle between the coordinate curves.

For orthogonal coordinates, associating γ_1 with ϕ and γ_2 with θ, the surface is referred to lines of curvature coordinates, since the lines of curvature of a surface of revolution are its meridians and parallels. Hence, the quantities r_1' and r_2' become r_1 and r_2, i.e., the principal radii of curvature, and

$$A = r_1, \quad B = r, \quad \Omega = 0 \tag{71a}$$

Then,

$$(ds)^2 = r_1^2(d\phi)^2 + r^2(d\theta)^2 \tag{70a}$$

Comparing the Equation (67) with Equation (69)

$$1 - \nu^2 = \frac{S_{11}S_{22} - S_{12}^2}{S_{11}^2}$$

and (72)

$$E = \frac{1}{S_{11}}$$

Therefore, the stresses and deformations in orthotropic shells can be calculated from the Gol'denveizer's edge effect solutions, within its range of validity, by means of the above relationships.

The membrane state of stress is determined by the quantities:

$$N_1^m, N_2^m, N_{12}^m, N_{21}^m, u^m, v^m, w^m, \chi_1^m$$

and the edge effect by the quantities:

$$N_2^*, M_1^*, M_2^*, w^*, \chi_1^*, Q_1^*$$

Hence, the stress resultants and displacements of the complete state of stress may be written in the form:

$$\begin{array}{llll}
N_1 = N_1^m ; & M_1 = M_1^* ; & u = u^m & \\
N_2 = N_2^m + N_2^* ; & M_2 = M_2^* ; & v = v^m & \\
N_{12} = N_{12}^m ; & M_{12} = M_{21} = 0 ; & w = w^m + w^* & \quad (73) \\
N_{21} = N_{21}^m ; & Q_1 = Q_1^* ; & \chi_1 = \chi_1^m + \chi_1^* & \\
 & Q_2 = 0 & &
\end{array}$$

The formulas for the edge effect theory are outlined below. Here, and in the following pages, the subscript 0 indicates the edge (Figure 12).

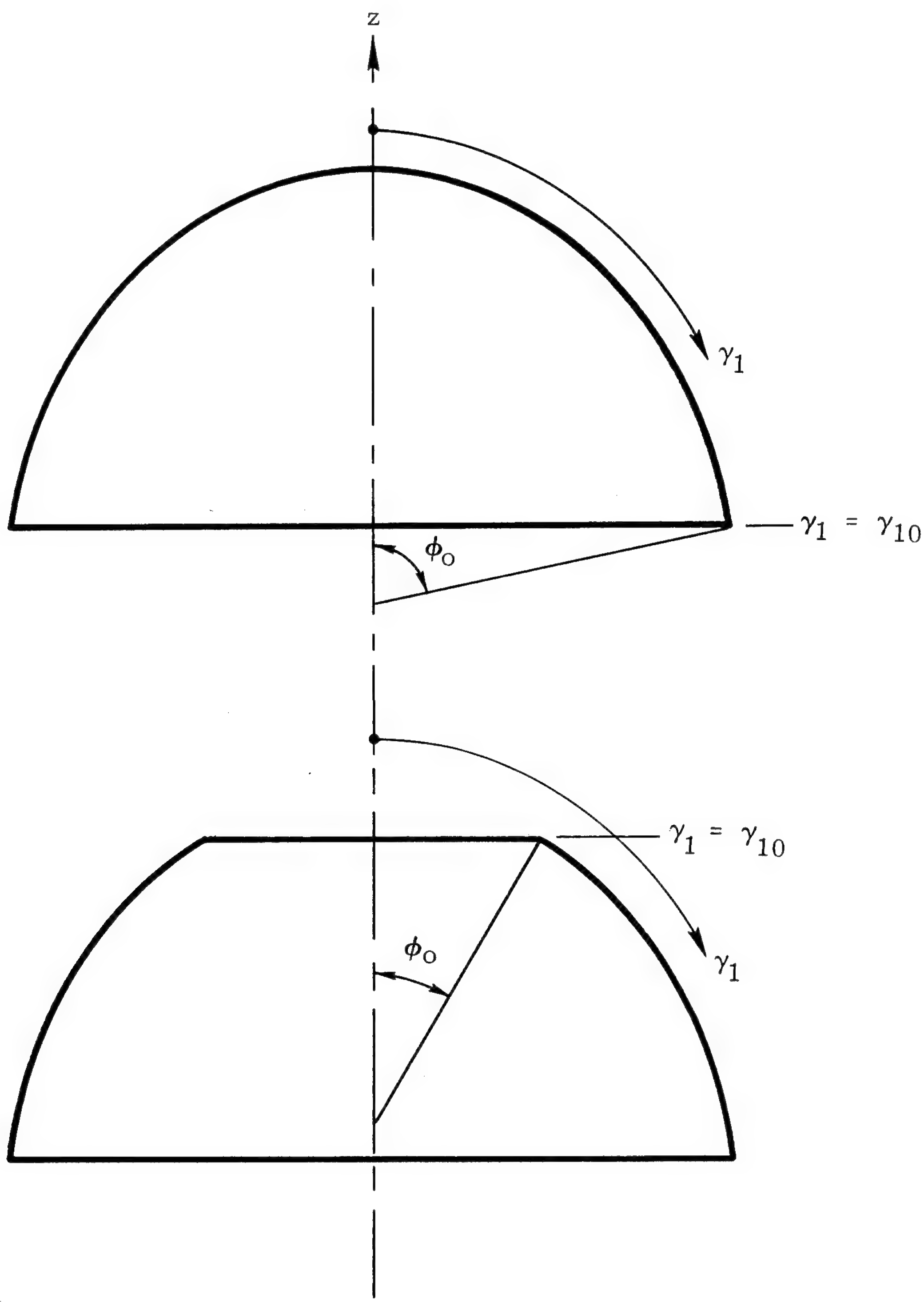


Figure 12. Lower and Upper Edges of
a Shell of Revolution

1) For the lower edge, $\gamma_1 \leqq \gamma_{10}$

$$Ehw^* = \left[\Gamma_1 \cos kg(\gamma_1 - \gamma_{10}) + \Gamma_2 \sin kg(\gamma_1 - \gamma_{10})\right] e^{kg(\gamma_1 - \gamma_{10})}$$

$$Eh\chi_1^* = -\frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{hr_2'}} \left\{\left[\Gamma_1 + \Gamma_2\right] \cos kg(\gamma_1 - \gamma_{10}) - \left[\Gamma_1 + \Gamma_2\right] \sin kg(\gamma_1 - \gamma_{10})\right\}$$
$$\times\ e^{kg(\gamma_1 - \gamma_{10})}$$

$$M_1^* = \frac{1}{\nu} M_2^* = -\frac{h}{2r_2'\sqrt{3(1-\nu^2)}} \left[\Gamma_2 \cos kg(\gamma_1 - \gamma_{10}) - \Gamma_1 \sin kg(\gamma_1 - \gamma_{10})\right]$$
$$\times\ e^{kg(\gamma_1 - \gamma_{10})} \qquad (74)$$

$$Q_1^* = -\frac{1}{\sqrt[4]{12(1-\nu^2)}} \frac{1}{r_2'} \sqrt{\frac{h}{2r_2'}} \left\{-\left[\Gamma_1 - \Gamma_2\right] \cos kg(\gamma_1 - \gamma_{10}) - \left[\Gamma_1 + \Gamma_2\right]\right.$$
$$\left. \times\ \sin kg(\gamma_1 - \gamma_{10})\right\} e^{kg(\gamma_1 - \gamma_{10})}$$

$$N_2^* = -\frac{1}{r_2'} \left[\Gamma_1 \cos kg(\gamma_1 - \gamma_{10}) + \Gamma_2 \sin kg(\gamma_1 - \gamma_{10})\right] e^{kg(\gamma_1 - \gamma_{10})}$$

2) For the upper edge, $\gamma_1 \geqq \gamma_{10}$

$$Ehw^* = \left[\Gamma_3 \cos kg(\gamma_1 - \gamma_{10}) + \Gamma_4 \sin kg(\gamma_1 - \gamma_{10})\right] e^{-kg(\gamma_1 - \gamma_{10})}$$

$$Eh\chi_1^* = -\frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{hr_2'}} \left\{-\left[\Gamma_3 - \Gamma_4\right] \cos kg(\gamma_1 - \gamma_{10}) - \left[\Gamma_3 + \Gamma_4\right]\right. \qquad (75)$$
$$\left. \times\ \sin kg(\gamma_1 - \gamma_{10})\right\} e^{-kg(\gamma_1 - \gamma_{10})}$$

$$M_1^* = \frac{1}{\nu} M_2^* = - \frac{h}{2 r_2' \sqrt{3(1 - \nu^2)}} \left[- \Gamma_4 \cos kg (\gamma_1 - \gamma_{10}) + \Gamma_4 \sin kg (\gamma_1 - \gamma_{10}) \right]$$

$$\times \; e^{-kg(\gamma_1 - \gamma_{10})}$$

$$Q_1^* = - \frac{1}{\sqrt[4]{12(1 - \nu^2)}} \frac{1}{r_2'} \sqrt{\frac{h}{2 r_2'}} \left\{ \left[\Gamma_3 + \Gamma_4 \right] \cos kg (\gamma_1 - \gamma_{10}) - \left[\Gamma_3 - \Gamma_4 \right] \right. \tag{75}$$

$$\left. \times \; \sin kg (\gamma_1 - \gamma_{10}) \right\} e^{-kg(\gamma_1 - \gamma_{10})}$$

$$N_2^* = - \frac{1}{r_2'} \left[\Gamma_3 \cos kg (\gamma_1 - \gamma_{10}) + \Gamma_4 \sin kg (\gamma_1 - \gamma_{10}) \right] e^{-kg(\gamma_1 - \gamma_{10})}$$

where Γ_1, Γ_2, Γ_3, Γ_4 are arbitrary functions of γ_2

and

$$kg = \left[\frac{3(1 - \nu^2)}{4} \right]^{1/4} \frac{A_o}{\sqrt{\frac{1}{2} h r_{20}'}} \tag{76}$$

A_0 and r_{20}' are the values of the quantities A and r_2' at the edge of the shell.

The Edge Effect Near a Clamped Edge

1) Upper Edge ($\gamma_1 \geqq \gamma_{10}$ or $\phi \geqq \phi_0$)

The boundary conditions at the supported edge, $\gamma_1 = \gamma_{10}$ are

$$u = 0 ; \quad v = 0 ; \quad w = 0 ; \quad \chi_1 = 0 \tag{77}$$

The first two of these four conditions are tangential, the last two are non-tangential. The tangential boundary conditions have been fulfilled by the membrane theory, since by Expressions (73)

$$u_0 = u_0^m = 0 ; \quad v_0 = v_0^m = 0 \tag{78}$$

The non-tangential boundary conditions may be written in the form,

$$\begin{aligned} Ehw_0 &= Ehw_0^m + Ehw_0^* = 0 \\ Eh\chi_{10} &= Eh\chi_{10}^m + Eh\chi_{10}^* = 0 \end{aligned} \tag{79}$$

For $\gamma_1 \geqq \gamma_{10}$ using Equation (75)

$$Ehw_0^* = \Gamma_3$$

$$Eh\chi_{10}^* = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{hr'_{20}}}\left[\Gamma_3 - \Gamma_4\right]$$

Hence

$$\begin{aligned} \Gamma_3 &= - Ehw_0^m \\ \Gamma_4 &= - Ehw_0^m + \frac{\sqrt{hr'_{20}}}{\sqrt[4]{3(1-\nu^2)}} Eh\chi_{10}^m \end{aligned} \tag{79a}$$

Obviously, for the membrane analysis of the shell, w and $r'_{20}\chi_1$ are quantities of the same order, and on the right-hand side of the above expression, therefore, the principal part will be played by the first term, so that we may assume as an approximation that

$$\Gamma_4 = - Ehw_0^m \tag{79b}$$

Then, for the computation of the stress resultants and moments of the edge effect due to loads of rotational symmetry

$$N_2^* = -\frac{1}{r_2'}\left[\Gamma_3 \cos kg\,(\gamma_1 - \gamma_{10}) + \Gamma_4 \sin kg\,(\gamma_1 - \gamma_{10})\right] e^{-kg(\gamma_1 - \gamma_{10})}$$

$$M_1^* = \frac{M_2^*}{\nu} = -\frac{h}{2r_2'\sqrt{3(1-\nu^2)}}\left[-\Gamma_4 \cos kg\,(\gamma_1 - \gamma_{10}) + \Gamma_3 \sin kg\,(\gamma_1 - \gamma_{10})\right] \tag{80}$$

$$\times\, e^{-kg(\gamma_1 - \gamma_{10})}$$

A_0 and r_{20}' are the values of the quantities A and r_2' on the boundary of the edge.

Substituting for Γ_3 and Γ_4 in Equation (80)

$$N_2^* = -\frac{1}{r_2'}\left[-Ehw_0^m \cos kg\,(\gamma_1 - \gamma_{10}) - Ehw_0^m \sin kg\,(\gamma_1 - \gamma_{10})\right] e^{-kg(\gamma_1 - \gamma_{10})}$$

$$= \frac{Ehw_0^m}{r_2'}\left[\cos kg\,(\gamma_1 - \gamma_{10}) + \sin kg\,(\gamma_1 - \gamma_{10})\right] e^{-kg(\gamma_1 - \gamma_{10})}$$

$$M_1^* = \frac{M_2^*}{\nu} = -\frac{h}{2r_2'\sqrt{3(1-\nu^2)}}\left[Ehw_0^m \cos kg\,(\gamma_1 - \gamma_{10}) - Ehw_0^m \sin kg\,(\gamma_1 - \gamma_{10})\right] \tag{81}$$

$$\times\, e^{-kg(\gamma_1 - \gamma_{10})}$$

$$= -\frac{h}{2r_2'}\frac{Ehw_0^m}{\sqrt{3(1-\nu^2)}}\left[\cos kg\,(\gamma_1 - \gamma_{10}) - \sin kg\,(\gamma_1 - \gamma_{10})\right] e^{-kg(\gamma_1 - \gamma_{10})}$$

To eliminate w_0^m observe (Ref. 1)

$$\epsilon_2^m = \frac{1}{B}\frac{\partial v^m}{\partial \gamma_2} + \frac{1}{AB}\frac{\partial B}{\partial \gamma_1}u^m - \frac{w^m}{r_2'} = \frac{1}{Eh}\left(N_2^m - \nu N_1^m\right)$$

Since at $\gamma_1 = \gamma_{10}$

$$u^m = v^m = 0$$

$$w_0^m = -\frac{r'_{20}}{Eh}\left(N_{20}^m - \nu N_{10}^m\right)$$

Thus, Equation (81) may be written

$$N_2^* = -\frac{r'_{20}}{r'_2}\left(N_{20}^m - \nu N_{10}^m\right)\left[\cos kg(\gamma_1 - \gamma_{10}) + \sin kg(\gamma_1 - \gamma_{10})\right] e^{-kg(\gamma_1 - \gamma_{10})}$$

$$M_1^* = \frac{M_2^*}{\nu} = \frac{h}{2r'_2}\,\frac{r'_{20}\left(N_{20}^m - \nu N_{10}^m\right)}{\sqrt{3(1-\nu^2)}}\left[\cos kg(\gamma_1 - \gamma_{10}) - \sin kg(\gamma_1 - \gamma_{10})\right] \times e^{-kg(\gamma_1 - \gamma_{10})} \qquad (82)$$

In the above formulas, the quantity $\dfrac{r'_{20}}{r'_2}$ should be set equal to unity, because such an assumption had been adopted by Gol'denveizer in the general formulation of the theory.

By choosing $\gamma_1 = \phi$ and $\gamma_2 = \theta$, the surface is referred to lines of curvature coordinates, since the lines of curvature of a surface of revolution are its meridians and parallels.

Hence, the quantities r'_1 and r'_2 become r_1 and r_2, i.e., the principal radii of curvature.

Rewriting Equation (82) in the notation of the membrane theory presented earlier in this report, the following expression Equation (83) for the upper edge, $\phi \geqq \phi_0$, are obtained.

$$N_\theta^* = -\left[N_\theta^m - \nu N_\phi^m\right]_{\phi=\phi_0} \left[\cos\lambda(\phi - \phi_0) + \sin\lambda(\phi - \phi_0)\right] e^{-\lambda(\phi-\phi_0)}$$

$$M_\theta^* = \frac{M_\theta^*}{\nu} = \frac{h\left[N_\theta^m - \nu N_\phi^m\right]_{\phi=\phi_0}}{2\sqrt{3(1-\nu^2)}} \left[\cos\lambda(\phi - \phi_0) - \sin\lambda(\phi - \phi_0)\right] e^{-\lambda(\phi-\phi_0)} \tag{83}$$

where

$$\lambda = \sqrt[4]{\frac{3(1-\nu^2)}{4}} \frac{r_{10}}{\sqrt{\frac{1}{2} h r_{20}}} \tag{84}$$

2) Lower Edge, $(\gamma \leqq \gamma_{10}$ or $\phi \leqq \phi_0)$

$$N_\theta^* = -\left[N_\theta^m - \nu N_\phi^m\right]_{\phi=\phi_0} \left[\cos\lambda(\phi - \phi_0) - \sin\lambda(\phi - \phi_0)\right] e^{\lambda(\phi-\phi_0)}$$

$$M_\phi^* = \frac{M_\theta^*}{\nu} = \frac{h\left[N_\theta^m - \nu N_\phi^m\right]_{\phi=\phi_0}}{2\sqrt{3(1-\nu^2)}} \left[\cos\lambda(\phi - \phi_0) + \sin\lambda(\phi - \phi_0)\right] e^{\lambda(\phi-\phi_0)} \tag{85}$$

For spherical shells, the position vector,

$$\vec{r}_0 = R\sin\phi\cos\theta\, \underset{\sim}{e}_x + R\sin\phi\sin\theta\, \underset{\sim}{e}_y + R\cos\phi\, \underset{\sim}{e}_z$$

where $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, $\underset{\sim}{e}_z$ are unit vectors along the coordinate axes x, y, z, and $R = |\vec{r}_0|$.

Associating γ_1 with ϕ and γ_2 with θ

$$\frac{\partial \vec{r}_0}{\partial \gamma_1} = \frac{\partial \vec{r}_0}{\partial \phi} = R \cos \phi \cos \theta \, \underset{\sim}{e}_x + R \cos \phi \sin \theta \, \underset{\sim}{e}_y - R \sin \phi \, \underset{\sim}{e}_z$$

$$A = \left| \frac{\partial \vec{r}_0}{\partial \phi} \right| = R$$

and similarly $B = r = R \sin \phi$.

Since $r_1 = r_2 = R$

$$\lambda = \frac{1}{2} \sqrt[4]{12(1 - \nu^2)} \sqrt{\frac{2R}{h}} \tag{84a}$$

The Edge Effect Near a Hinged Supported Edge

1) Upper Edge ($\gamma_1 \geqq \gamma_{10}$ or $\phi \geqq \phi_0$)

The boundary conditions:

$$N_1 = 0 \, ; \qquad v = 0 \text{ (tangential)}$$

$$w = 0 \, ; \qquad M_1 = 0 \text{ (non-tangential)} \tag{86}$$

$$Ehw_0 = Ehw_0^m + Ehw_0^* = 0$$

$$M_1^* = 0$$

From Equation (75), at $\gamma_1 = \gamma_{10}$

$$\Gamma_3(\gamma_2) = -Ehw_0^m \, ; \qquad \Gamma_4(\gamma_2) = 0$$

Substituting in Equation (80) and changing the notation,

$$N_\theta^* = \frac{Ehw_0^m}{r_2} e^{-\lambda(\phi - \phi_0)} \cos\lambda(\phi - \phi_0)$$

$$M_\phi^* = \frac{1}{\nu} M_\theta^* = \frac{Eh^2 w_0^m}{2r_2\sqrt{3(1-\nu^2)}} e^{-\lambda(\phi-\phi_0)} \sin\lambda(\phi - \phi_0) \tag{87}$$

where λ is given by expression (84a)

2) Lower Edge, $(\gamma_1 \leqq \gamma_{10}$ or $\phi \leqq \phi_0)$

$$N_\theta^* = \frac{Ehw_0^m}{r_2} e^{\lambda(\phi - \phi_0)} \cos\lambda(\phi - \phi_0)$$

$$M_\phi^* = \frac{1}{\nu} M_\theta^* = -\frac{Eh^2 w_0^m}{2r_2\sqrt{3(1-\nu^2)}} e^{\lambda(\phi-\phi_0)} \sin\lambda(\phi - \phi_0) \tag{88}$$

where λ is given by Equation (84a).

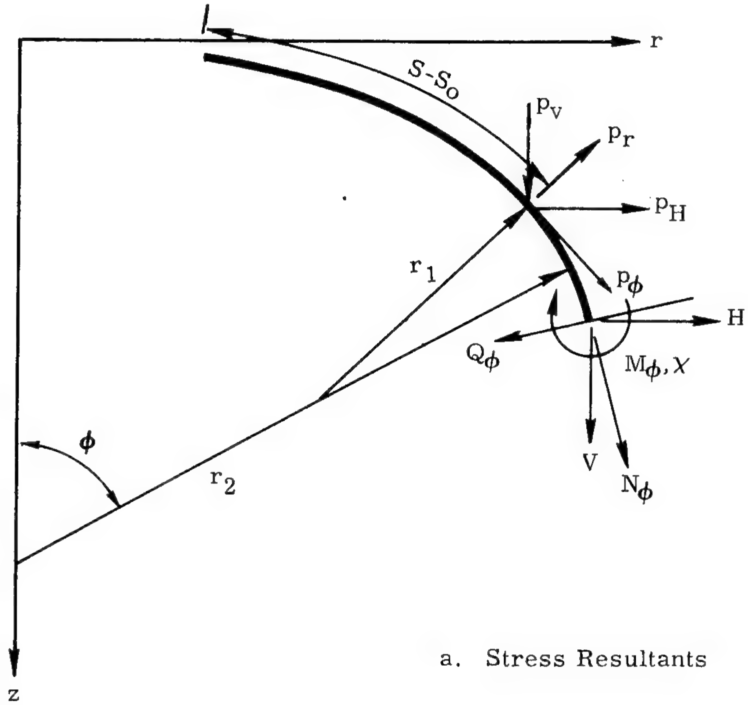
B. Asymptotic Solution

The equilibrium equations for the case of rotational symmetry (Ref. 2),

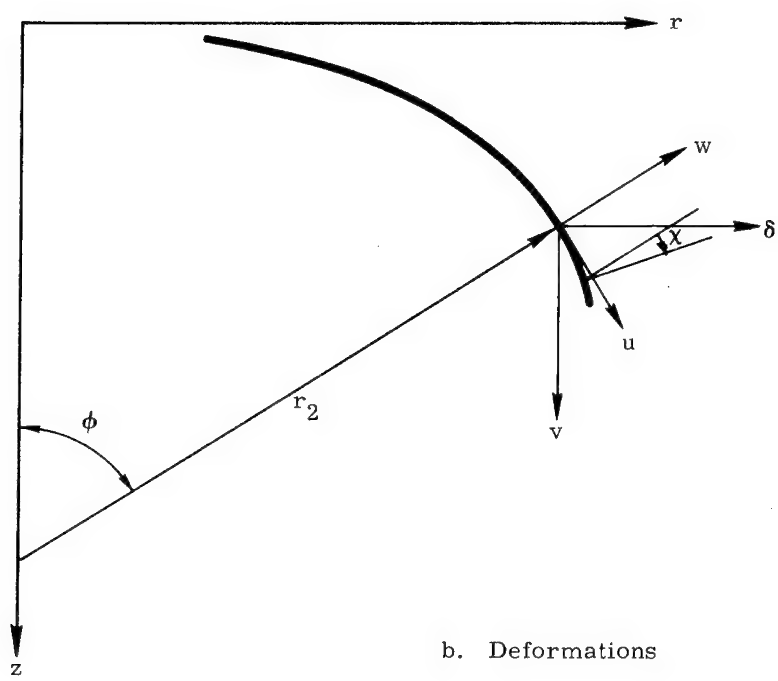
$$\frac{Eh^3}{12(1-\nu^2)}\left[\frac{d}{ds}\left(r\frac{d\chi}{ds}\right) - \left(\nu\frac{\sin\phi}{r_1} + \frac{\cos^2\phi}{r}\right)\chi\right] - rH\sin\phi = -rV\cos\phi$$

$$\frac{1}{Eh}\left[\frac{d}{ds}\left(r\frac{d}{ds}(rH)\right) + \left(\nu\frac{\sin\phi}{r_1} - \frac{\cos^2\phi}{r}\right)rH\right] + \chi\sin\phi = \frac{1}{Eh}\left[-\frac{d}{ds}(r^2 p_H - \nu rV\sin\phi) + \cos\phi\,(V\sin\phi - \nu r p_H)\right] \tag{89}$$

where, χ is the rotation of the normal to the shell and V and H are the axial and radial components of the stress resultants (Figure 13), respectively.



a. Stress Resultants



b. Deformations

Figure 13. Shell Geometry

The axial and radial components of the stress resultants,

$$V = N_\phi \sin\phi - Q_\phi \cos\phi$$
$$H = N_\phi \cos\phi + Q_\phi \sin\phi \qquad (90)$$

Similarly, the deflection of a point on the midsurface can be given by the radial component δ and the axial component v,

$$\delta = w \sin\phi + u \cos\phi$$
$$\tilde{v} = -w \cos\phi + u \sin\phi \qquad (91)$$

For homogeneous equations,

$$p_r = p_\phi = p_v = p_H \equiv 0$$

Introducing a non-dimensional quantity,

$$\mathcal{H} = \left[12(1-\nu^2)\right]^{1/2} \frac{rH}{Eh^2}$$

the homogeneous equations are transformed into the form (Ref. 2)

$$\frac{d}{ds}\left(r\frac{d\chi}{ds}\right) - \left(\nu \frac{\sin\phi}{r_1} + \frac{\cos^2\phi}{r}\right)\chi - \frac{\sqrt{12(1-\nu^2)}}{h}\mathcal{H}\sin\phi = 0$$
$$\frac{d}{ds}\left(r\frac{d\mathcal{H}}{ds}\right) + \left(\nu \frac{\sin\phi}{r_1} - \frac{\cos^2\phi}{r}\right)\mathcal{H} + \frac{\sqrt{12(1-\nu^2)}}{h}\chi\sin\phi = 0 \qquad (89a)$$

Assuming a solution in the form of a series expansion

$$\mathcal{H} = e^{\xi h^{-1/2}}\left[\mathcal{H}_0 + h^{1/2}\mathcal{H}_1 + h\mathcal{H}_2 + ---\right]$$
$$\chi = e^{\xi h^{-1/2}}\left[\chi_0 + h^{1/2}\chi_1 + h\chi_2 + ---\right] \qquad (92)$$

where ξ, χ_j, $\mathcal{H}_j$ are bounded functions of s that are completely independent of h.

Substituting into the homogeneous equations and equating the coefficients of h's equal to zero and retaining the leading terms in the expansion, the solution becomes

$$\chi = \operatorname{Re} \tilde{C} e^{\pm(1 + i)\xi} \left[1 + O\left((h/r)^{1/2}\right)\right] \tag{93}$$

where $\tilde{C}$ is a complex constant

$$\xi = \zeta - \zeta_0 = \left[3(1 - \nu^2)\right]^{1/4} \left[\frac{(s - s_0)}{\sqrt{hr_{20}}}\right] \tag{94}$$

and $r_{20} = (r_2)_{edge}$

For thin shell, r_1, $r_2 >> h$, say $\frac{r_1}{h}$, $\frac{r_2}{h} \gtrsim 20$ for the simple edge effect solution to be valid

$$\left[\frac{12(1 - \nu^2)r^2 \sin^2\phi}{h^2}\right]^{1/4} \left[\frac{1}{\left(1 + \frac{r_2}{r_1}\right) \cos\phi}\right] \gtrsim 5 \tag{95}$$

For sphere, $r = R \sin\phi$, $\quad \frac{r_2}{r_1} = 1$

$$\left[\frac{12(1 - \nu^2)R^2}{h^2}\right]^{1/4} \tan\phi \gtrsim 10 \tag{96a}$$

For pressure vessels, simple exponential "steep shell" solution gives valid results for

$$\left[\frac{12(1 - \nu^2)R^2}{h^2}\right]^{1/4} \tan\phi \gtrsim 3 \tag{96b}$$

Stress Resultants and Displacements for Upper Edge

$$\chi = (C_1 \cos \xi + C_2 \sin \xi) e^{-\xi} \tag{93a}$$

$$M_\phi^* = \frac{Eh^3}{12(1 - \nu^2)} \left[\frac{d\chi}{ds} + \nu \chi \frac{\cos \phi}{r} \right] \tag{97}$$

$$M_\phi^* \approx \frac{Eh^3}{12(1 - \nu^2)} \xi' \cdot \frac{d\chi}{d\xi} \tag{97a}$$

where

$$\xi' = \frac{d\xi}{ds}$$

$$M_\theta^* = \nu M_\phi^*$$

In the absence of surface loads, the horizontal equilibrium is,

$$N_\theta^* = \frac{d}{ds}(rH) \tag{98}$$

With

$$rH = \frac{Eh}{4r_2(\xi')^2} \frac{d^2\chi}{d\xi^2}$$

(98a)

$$N_\theta^* = \frac{Eh}{4r_2\xi'} \frac{d^3\chi}{d\xi^3}$$

Since

$$N_\phi^* = N_\theta^* O(\sqrt{h/r}) \; ; \qquad \frac{\delta^*}{r} \approx \frac{N_\theta^*}{Eh}$$

(where O denotes order of magnitude)

the Equations (97a) and (98a) become

$$
\begin{aligned}
M_{\phi}^{*} &= \frac{Eh^{3}}{12(1-\nu^{2})} \xi' \left[(C_2 - C_1)\cos\xi - (C_1 + C_2)\sin\xi\right] e^{-\xi} \\
N_{\theta}^{*} &= \frac{Eh}{2r_2\xi'} \left[(C_1 + C_2)\cos\xi + (C_2 - C_1)\sin\xi\right] e^{-\xi}
\end{aligned} \qquad (99)
$$

Membrane Deformations

Horizontal deformation:

$$\frac{\delta^{m}}{r} = \epsilon_{\theta} = \frac{1}{Eh}\left(N_{\theta}^{m} - \nu N_{\phi}^{m}\right) \qquad (100)$$

for uniformly distributed loading, p

$$\frac{\delta^{m}}{r} = \frac{pr_2}{2Eh}\left[2 - \frac{r_2}{r_1} - \nu\right]$$

$$\chi^{m} = \frac{pr\cos\phi}{2Eh\sin^{2}\phi}\left\{\frac{r}{\cos\phi}\frac{d}{ds}\left(\frac{r_2}{r_1}\right) - \left(2 - \frac{r_2}{r_1}\right)^{2} + 1\right\} \qquad (101)$$

Boundary Conditions for Fixed Support

$$
\begin{aligned}
\delta^{m} + \delta^{*} &= 0 \\
\chi^{m} + \chi^{*} &= 0
\end{aligned} \qquad (102)
$$

yields

$$
\begin{aligned}
\frac{\delta^{*}}{r} &= \frac{1}{2r_2\xi'} e^{-\xi}\left[(C_1 + C_2)\cos\xi + (C_2 - C_1)\sin\xi\right] \\
\chi^{*} &= (C_1\cos\xi + C_2\sin\xi)\, e^{-\xi}
\end{aligned} \qquad (102a)
$$

At the support

$$\frac{\delta_0^*}{r_0} = \frac{1}{2r_{20}\xi'}(C_1 + C_2) = -\frac{\delta_0^m}{r_0}$$

$$\chi_0^* = C_1 = -\chi_0^m$$

Hence,

$$C_2 = -\frac{2\delta_0^m r_{20}}{r_0}\xi' + \chi_0^m$$

Substituting in the Equation (99)

$$M_\phi^* = \frac{Eh^3}{12(1-\nu^2)}\xi'\left[\left(-\frac{2\delta_0^m r_{20}}{r_0}\xi' + 2\chi_0^m\right)\cos\xi + \left(\frac{2\delta_0^m r_{20}}{r_0}\xi'\right)\sin\xi\right]e^{-\xi}$$

$$N_\theta^* = \frac{Eh}{2r_2\xi'}\left[\left(-\frac{2\delta_0^m r_{20}}{r_0}\xi'\right)\cos\xi + \left(-\frac{2\delta_0^m r_{20}}{r_0}\xi' + 2\chi_0^m\right)\sin\xi\right]e^{-\xi} \tag{103}$$

Since for membrane theory δ_0^m and $r_{20}\chi^m$ are of the same order of magnitude, the expressions for M_ϕ^* and N_θ^* can be reduced to the following for general loading of a shell of revolution,

$$M_\phi^* = -\frac{Eh^3}{6(1-\nu^2)}(\xi')^2\left(\frac{\delta_0^m r_{20}}{r_0}\right)\left[\cos\xi - \sin\xi\right]e^{-\xi}$$

$$N_\theta^* = -\frac{Eh}{r_2}\frac{\delta_0^m r_{20}}{r_0}\left[\cos\xi + \sin\xi\right]e^{-\xi} \tag{103a}$$

Rewriting,

$$M_\phi^* = \frac{M_\theta^*}{\nu} = -\frac{h\left[N_\theta^m - \nu N_\phi^m\right]_{\zeta=\zeta_0}}{2\sqrt{3(1-\nu^2)}} \left\{\cos(\zeta - \zeta_0) - \sin(\zeta - \zeta_0)\right\} e^{-(\zeta - \zeta_0)}$$

$$N_\theta^* = -\frac{r_{20}}{r_2}\left[N_\theta^m - \nu N_\phi^m\right]_{\zeta=\zeta_0} \left\{\cos(\zeta - \zeta_0) + \sin(\zeta - \zeta_0)\right\} e^{-(\zeta - \zeta_0)} \tag{104}$$

Gol'denveizer had adopted the assumption: $\frac{r_{20}}{r_2} \approx 1$ in his formulation. Hence a comparison of the Gol'denveizer and asymtotic solutions reveals that the two methods for "Edge Effect" theory yield identical results.

For spherical shells,

$$\xi = \zeta - \zeta_0 = \left[3(1-\nu^2)\right]^{1/4} (\phi - \phi_0)\sqrt{\frac{R}{h}} \tag{94a}$$

Deformations for a Shell of Revolution Under Symmetric Loading

Membrane Deformations

$$\epsilon_\theta = S_{11}\sigma_\theta + S_{12}\sigma_\phi$$

$$\epsilon_\phi = S_{12}\sigma_\theta + S_{22}\sigma_\phi \tag{105}$$

$$\epsilon_\phi = \frac{1}{r_1}\left(\frac{du}{d\phi} + w\right)$$

$$\epsilon_\theta = \frac{1}{r_2}\left(u \cot\phi + w\right) \tag{57}$$

Equating Equations (57) and (105)

$$\frac{1}{r_2}(u \cot\phi + w) = S_{11}\sigma_\theta + S_{12}\sigma_\phi \tag{106a}$$

$$\frac{1}{r_1}\left(\frac{du}{d\phi} + w\right) = S_{12}\sigma_\theta + S_{22}\sigma_\phi \tag{107b}$$

Eliminating w, the differential equation for u in terms of stress resultants is

$$\frac{du}{d\phi} - u\cot\phi = \frac{1}{h}\left[N_\phi(r_1S_{22} - r_2S_{12}) + N_\theta\ (r_1S_{12} - r_2S_{11})\right]$$

which is the same as Equation (61) on Page 42.

This equation is in the form

$$\frac{du}{d\phi} + \Xi(\phi)\cdot u + \Upsilon(\phi) = 0 \tag{108}$$

which has the general solution

$$u = \left[C - \int \Upsilon e^{\int \Xi d\phi} d\phi\right] e^{-\int \Xi d\phi} \tag{109}$$

where

$$\Xi = -\cot\phi$$

$$\int \Xi d\phi = -\int \cot\phi\, d\phi$$

$$\Upsilon(\phi) = -\frac{N_\phi}{h}(r_1S_{22} - r_2S_{12}) - \frac{N_\theta}{h}(r_1S_{12} - r_2S_{11})$$

Then,

$$u = \left\{C - \int \Upsilon(\phi) e^{-\int \cot\phi\, d\phi} d\phi\right\} e^{\int \cot\phi\, d\phi} \tag{109a}$$

After integrating the simple functions,

$$u = \left\{C - \int \frac{\Upsilon(\phi)}{\sin\phi} d\phi\right\} \sin\phi \tag{109b}$$

Hence,

$$u = \left\{\int \frac{1}{h}\left[N_\phi(r_1S_{22} - r_2S_{12}) + N_\theta(r_1S_{12} - r_2S_{11})\right] \frac{d\phi}{\sin\phi} + C\right\} \sin\phi$$

$$w = -u\cot\phi + \frac{r_2}{h}(S_{11}N_\theta + S_{12}N_\phi) \tag{110}$$

For a uniformly symmetrical distributed loading, p

$$u = \left\{ \int \frac{pr_1 r_2}{2h} \left[S_{22} + 2\left(1 - \frac{r_2}{r_1}\right) S_{12} - \frac{r_2}{r_1}\left(2 - \frac{r_2}{r_1}\right) S_{11} \right] \frac{d\phi}{\sin\phi} + C \right\} \sin\phi$$

Similarly, (111)

$$w = -u \cot\phi + \frac{pr_2^2}{2h}\left[\left(2 - \frac{r_2}{r_1}\right) S_{11} + S_{12}\right]$$

Specializing for spherical shells (Figure 14)

$$r_1 = r_2 = R$$

$$N_\phi = N_\theta = N$$

$$u = \left\{ C + \frac{NR}{h} \int_{\phi_0}^{\phi} \frac{(S_{22} - S_{11})}{\sin\phi} d\phi \right\} \sin\phi$$

(112)

$$w = -u \cot\phi + \frac{NR}{h}(S_{11} + S_{12})$$

where, the moduli of compliance S_{11}, S_{12} and S_{22} are functions of ϕ.

This is the consequence of the variation of the filament pattern along the meridian.

At the juncture with the insert, the boundary of the shell requires

$$u = 0 \ @ \ \phi = \phi_0$$

Therefore,

$$u = \frac{NR}{h} \sin\phi \int_{\phi_0}^{\phi} \frac{(S_{22} - S_{11})}{\sin\phi} d\phi$$

(112a)

$$w = \frac{NR}{h}\left\{ S_{11} + S_{12} - \cos\phi \int_{\phi_0}^{\phi} \frac{(S_{22} - S_{11})}{\sin\phi} d\phi \right\}$$

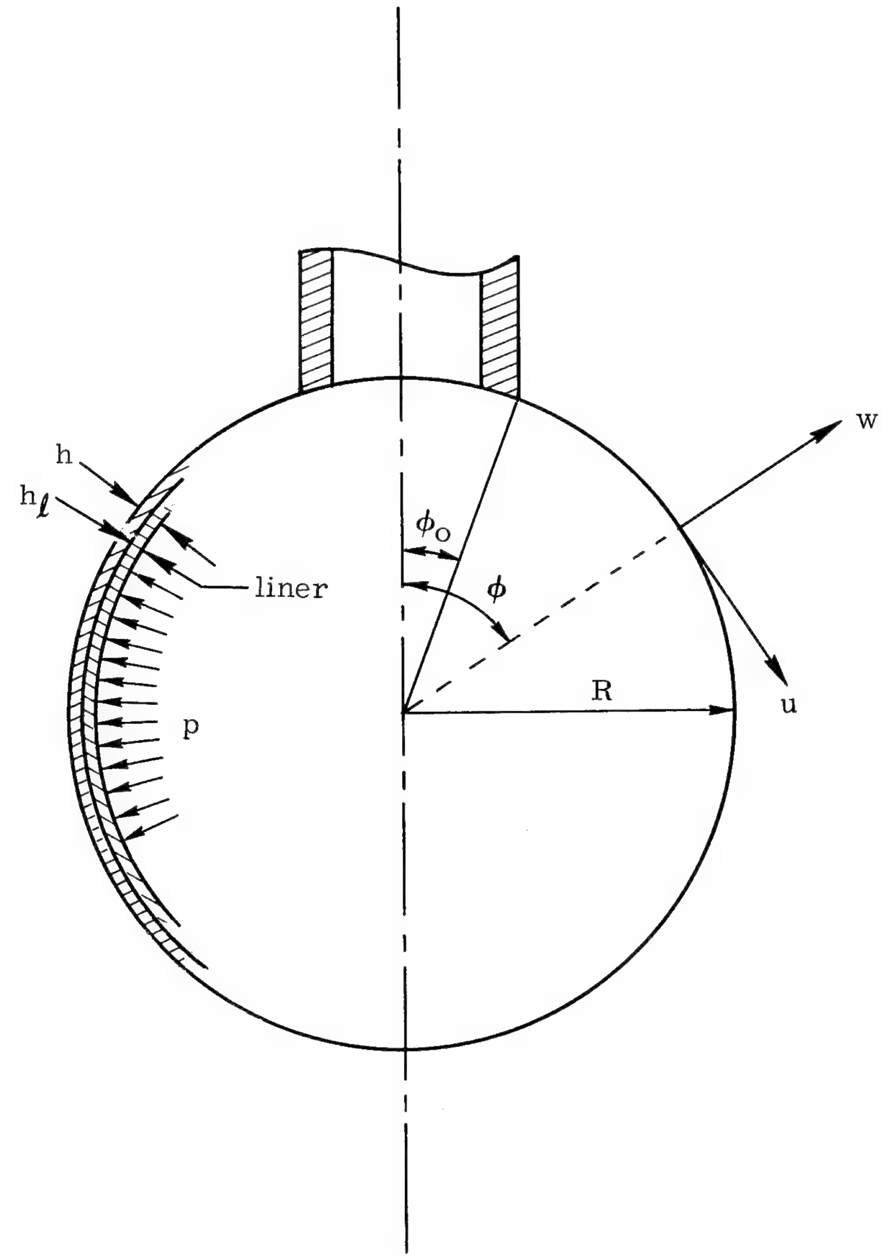


Figure 14. Spherical Pressure Vessel

Edge Effect Deformation of a Shell of Revolution for an Upper Edge

In what follows, the asymptotic solution (Ref. 2) is used

$$w^* = -\frac{1}{\xi'}\int \chi^* d\xi$$

$$= -\frac{1}{\xi'}\int [C_1 \cos\xi + C_2 \sin\xi] e^{-\xi} d\xi$$

$$= -\frac{e^{-\xi}}{2\xi'}[C_1(\sin\xi - \cos\xi) - C_2(\sin\xi + \cos\xi)] + C_3$$

at $\zeta = \zeta_0$

$$w_0^* = -w_0^m$$

By the same argument on the order of magnitudes of the terms,

$$-w_0^m = -\frac{1}{2\xi'}[-C_1 - C_2] + C_3 = -\frac{\delta_0^m r_{20}}{r_0} + C_3 = -\frac{\delta_0^m}{\sin\phi}$$

which yields $C_3 = 0$

Hence,

$$w^* = -\frac{1}{Eh}\left[N_\theta^m - \nu N_\phi^m\right]_{\zeta=\zeta_0} r_{20} e^{-\xi}(\sin\xi + \cos\xi)$$

$$w^* = -\frac{r_{20}}{Eh}\left[N_\theta^m - \nu N_\phi^m\right]_{\zeta=\zeta_0} \{\sin(\zeta - \zeta_0) + \cos(\zeta - \zeta_0)\} e^{-(\zeta-\zeta_0)}$$

Similarly, (113)

$$\chi^* = -\frac{2r_{20}}{Eh}\frac{[3(1-\nu^2)]^{1/4}}{\sqrt{hr_{20}}}\left(N_\theta^m - \nu N_\phi^m\right)_{\zeta=\zeta_0} e^{-(\zeta-\zeta_0)}\sin(\zeta - \zeta_0)$$

$$\xi = \zeta - \zeta_0 = \left[3(1 - \nu^2)\right]^{1/4} (\phi - \phi_0)\sqrt{\frac{R}{h}} \tag{94}$$

Total deformations will then be

$$w = w^* + w^m$$

$$u = u^m$$

$$\chi = \chi^* + \chi^m$$

APPENDIX -- STRESS TRANSFORMATIONS

Transformations Relations for Anisotropic Stresses

Under a rotation of axes from x, y, z to ξ, η, ζ, the stress components transform in accordance with the following relations

$$\sigma_{p,q} = \sum_{i}\sum_{j} \ell_{ip}\ell_{jq}\sigma_{ij} \qquad \text{A-1}$$

where

$p, q = \xi, \eta, \zeta$

$i, j = x, y, z$

ℓ_{pg} = direction cosines

For any coordinate axes:

	x	y	z
ξ	$\ell_{\xi x}$	$\ell_{\xi y}$	$\ell_{\xi z}$
η	$\ell_{\eta x}$	$\ell_{\eta y}$	$\ell_{\eta z}$
ζ	$\ell_{\zeta x}$	$\ell_{\zeta y}$	$\ell_{\zeta z}$

Expanding the expression (A-1) for $p = \xi$ and $q = \xi$

$$\sigma_{\xi\xi} = \ell^2_{x\xi}\sigma_{xx} + \ell^2_{y\xi}\sigma_{yy} + \ell^2_{z\xi}\sigma_{zz} + 2\ell_{x\xi}\ell_{y\xi}\sigma_{xy} + 2\ell_{x\xi}\ell_{z\xi}\sigma_{xz} + 2\ell_{y\xi}\ell_{z\xi}\sigma_{yz} \qquad \text{A-2}$$

Similarly

$$\sigma_{\eta\eta} = \ell^2_{x\eta}\sigma_{xx} + \ell^2_{y\eta}\sigma_{yy} + \ell^2_{z\eta}\sigma_{zz} + 2\ell_{x\eta}\ell_{y\eta}\sigma_{xy} + 2\ell_{x\eta}\ell_{z\eta}\sigma_{xz} + 2\ell_{y\eta}\ell_{z\eta}\sigma_{yz} \qquad \text{A-3}$$

$$\sigma_{\zeta\zeta} = \ell^2_{x\zeta}\sigma_{xx} + \ell^2_{y\zeta}\sigma_{yy} + \ell^2_{z\zeta}\sigma_{zz} + 2\ell_{x\zeta}\ell_{y\zeta}\sigma_{xy} + 2\ell_{x\zeta}\ell_{z\zeta}\sigma_{yz} + 2\ell_{y\zeta}\ell_{z\zeta}\sigma_{yz} \qquad \text{A-4}$$

For $p = \xi$; $q = \eta$

$$\sigma_{\xi\eta} = \ell_{x\xi}\ell_{x\eta}\sigma_{xx} + \ell_{y\xi}\ell_{y\eta}\sigma_{yy} + \ell_{z\xi}\ell_{z\eta}\sigma_{zz} + \left(\ell_{x\xi}\ell_{y\eta} + \ell_{y\xi}\ell_{x\eta}\right)\sigma_{xy} + \left(\ell_{x\xi}\ell_{z\eta} + \ell_{z\xi}\ell_{x\eta}\right)\sigma_{xz} + \left(\ell_{y\xi}\ell_{z\eta} + \ell_{z\xi}\ell_{y\eta}\right)\sigma_{yz} \qquad \text{A-5}$$

For $p = \xi$; $q = \zeta$

$$\sigma_{\xi\zeta} = \ell_{x\xi}\ell_{x\zeta}\sigma_{xx} + \ell_{y\xi}\ell_{y\zeta}\sigma_{yy} + \ell_{z\xi}\ell_{z\zeta}\sigma_{zz} + \left(\ell_{x\xi}\ell_{y\zeta} + \ell_{y\xi}\ell_{x\zeta}\right)\sigma_{xy} + \left(\ell_{x\xi}\ell_{z\zeta} + \ell_{z\xi}\ell_{x\zeta}\right)\sigma_{xz} + \left(\ell_{y\xi}\ell_{z\zeta} + \ell_{z\xi}\ell_{y\zeta}\right)\sigma_{zy} \qquad \text{A-6}$$

For p = η; q = ζ

$$\sigma_{\eta\zeta} = \ell_{x\eta}\ell_{x\zeta}\sigma_{xx} + \ell_{y\eta}\ell_{y\zeta}\sigma_{yy} + \ell_{z\eta}\ell_{z\zeta}\sigma_{zz} + \left(\ell_{x\eta}\ell_{y\zeta} + \ell_{y\eta}\ell_{x\zeta}\right)\sigma_{xy} + \left(\ell_{x\eta}\ell_{z\zeta} + \ell_{z\eta}\ell_{x\zeta}\right)\sigma_{xz} + \left(\ell_{y\eta}\ell_{z\zeta} + \ell_{z\eta}\ell_{y\zeta}\right)\sigma_{yz} \qquad \text{A-7}$$

Stress Transformations for an Orthotropic Sheet

Analysis of an orthotropic sheet reduces to a two dimensional problem because the stress and strain components normal to the sheet surface, are assumed to vanish.

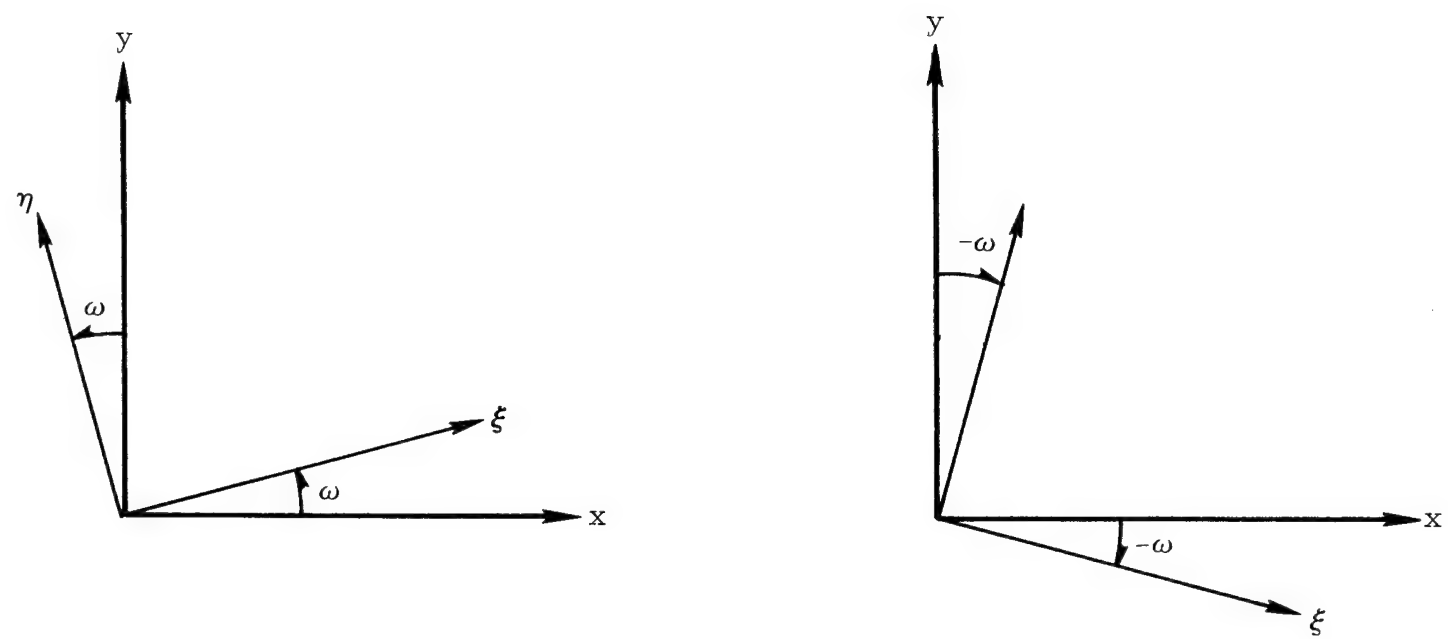
This is equivalent to rotation about the z-axis (Figure 15). Therefore,

	x	y	z
ξ	$\cos\omega$	$\cos\left(\frac{\pi}{2} - \omega\right) = \sin\omega$	0
η	$\cos\left(\frac{\pi}{2} + \omega\right) = -\sin\omega$	$\cos\omega$	0
ζ	0	0	1

$$\sigma_{\xi\xi} = \sigma_{xx}\cos^2\omega + \sigma_{yy}\sin^2\omega + 2\sigma_{xy}\cos\omega\sin\omega$$

$$\sigma_{\eta\eta} = \sigma_{xx}\sin^2\omega + \sigma_{yy}\cos^2\omega - 2\sigma_{xy}\sin\omega\cos\omega \qquad \text{A-8}$$

$$\sigma_{\xi\eta} = -\sigma_{xx}\cos\omega\sin\omega + \sigma_{yy}\sin\omega\cos\omega + \left(\cos^2\omega - \sin^2\omega\right)\sigma_{xy}$$



a. Counterclockwise Rotation

b. Clockwise Rotation

Figure 15. Rotation of the Coordinate System about the z Axis

Similarly, the strain transformations can be written,

$$
\begin{aligned}
\epsilon_{\xi\xi} &= \epsilon_{xx}\cos^2\omega + \epsilon_{yy}\sin^2\omega + \epsilon_{xy}\cos\omega\sin\omega \\
\epsilon_{\eta\eta} &= \epsilon_{xx}\sin^2\omega + \epsilon_{yy}\cos^2\omega + \epsilon_{xy}\cos\omega\sin\omega \qquad \text{A-9} \\
\epsilon_{\xi\eta} &= 2\left(\epsilon_{yy} - \epsilon_{xx}\right)\sin\omega\cos\omega + \epsilon_{xy}\left(\cos^2\omega - \sin^2\omega\right)
\end{aligned}
$$

	x	y	z
ξ	$\cos(-\omega) = \cos\omega$	$\cos\left(\frac{\pi}{2} + \omega\right) = -\sin\omega$	0
η	$\cos\left(\frac{\pi}{2} - \omega\right) = \sin\omega$	$\cos\omega$	0
ζ	0	0	1

$$
\begin{aligned}
\sigma_{\xi\xi} &= \sigma_{xx}\cos^2\omega + \sigma_{yy}\sin^2\omega - 2\sigma_{xy}\sin\omega\cos\omega \\
\sigma_{\eta\eta} &= \sigma_{xx}\sin^2\omega + \sigma_{yy}\cos^2\omega + 2\sigma_{xy}\sin\omega\cos\omega \qquad \text{A-10} \\
\sigma_{\xi\eta} &= \sigma_{xx}\sin\omega\cos\omega - \sigma_{yy}\sin\omega\cos\omega + \left(\cos^2\omega - \sin^2\omega\right)\sigma_{xy}
\end{aligned}
$$

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